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EXCLUDING BLOWUP AT ZERO POINTS OF THE POTENTIAL BY MEANS OF LIOUVILLE-TYPE THEOREMS

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ABSTRACT. We prove a local version of a (global) result of Merle and Zaag about ODE behavior of solutions near blowup points for subcritical nonlinear heat equations. As an application, for the equation $u_t = \Delta u + V(x)f(u)$, we rule out the possibility of blowup at zero points of the potential V for monotone in time solutions when $f(u) \sim u^p$ for large u, both in the Sobolev subcritical case and in the radial case. This solves a problem left open in previous work on the subject. Suitable Liouville-type theorems play a crucial role in the proofs.

1. INTRODUCTION

In this paper, we consider the following semilinear heat equation with spatially dependent coefficient in the nonlinearity:

$$u_t = \Delta u + V(x)f(u), \quad 0 < t < T,\ x \in \Omega. \tag{1.1}$$

In the case when V is a positive constant and $f(u) \sim u^p$ with $p > 1$, the blowup behavior of solutions has received considerable attention in the past decades and a rich variety of phenomena has been discovered (see, e.g., the monograph [30] and the references therein). In the case when the potential is nonnegative and nonconstant, it is a natural question whether or not blowup can occur at zero points of the potential V. Although the answer would intuitively seem to be negative at first sight, it was surprisingly found in [6, 14, 16, 15] to be positive or negative depending on the situation (see Remark 1.2 for details).

The goal of this paper is twofold:

(i) rule out the possibility of blowup at zero points of the potential V for monotone in time solutions of equation (1.1) when $f(u) \sim u^p$ for large u.

(ii) prove a local version of a (global) result of Merle-Zaag [23] about ODE behavior of solutions near blowup points for subcritical nonlinear heat equations. This result, of

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independent interest, and which seems to be new even in the case $V \equiv 1$, will be an essential ingredient for (i).

Let us now state our general assumptions:

$$\Omega \text{ is a domain of } \mathbb{R}^n, \tag{1.2}$$

$$f : [0,\infty) \to [0,\infty) \text{ is a function of class } C^1, \tag{1.3}$$

$$\lim_{s\to\infty} s^{-p} f(s) = 1 \text{ for some } p > 1, \tag{1.4}$$

$$|f'(s)| \le C(1 + s^{p-1}), \quad s \ge 0, \tag{1.5}$$

$$V : \overline{\Omega} \to [0,\infty) \text{ is a Hölder continuous function.} \tag{1.6}$$

Throughout this article, we set

$$p_S = (n+2)/(n-2)_+, \qquad \alpha = 1/(p-1), \qquad \kappa = \alpha^\alpha$$

and we denote the zeroset of V by

$$\mathcal{V}_0 = \bigl\{x \in \overline{\Omega};\ V(x) = 0\bigr\}.$$

Our first main result rules out the possibility of blowup at zero points of the potential V for monotone in time solutions of (1.1), under suitable assumptions. In fact, the case of the homogeneous Dirichlet problem associated with equation (1.1) with $f(0) = 0$ was completely solved in [16]. The more delicate case $f(0) > 0$ was left as an open problem. We here essentially solve it for subcritical p, under a mild geometric assumption. Actually, the result here is formulated in a completely local way, without reference to any boundary conditions. Here, x_0 is said to be a blowup point if $\limsup_{t\to T,\, x\to x_0} u(t,x) = \infty$.

Theorem 1.1. *Assume* (1.2)-(1.6)*, with Ω bounded, f of class C^2 and convex. Let $x_0 \in \Omega$ be such that $V(x_0) = 0$ and*

$$\textit{the connected component of } \mathcal{V}_0 \textit{ containing } x_0 \textit{ does not intersect } \partial\Omega. \tag{1.7}$$

(i) Assume that $p < p_S$ and let u be a nonnegative classical solution of (1.1) *such that $u_t \ge 0$. Then x_0 is not a blowup point of u.*

(ii) Assertion (i) remains valid for any $p > 1$, if we assume in addition that u and V are radially symmetric and $\Omega = B_R$.

We stress that the assumption $u_t \ge 0$ cannot be removed in general (compare Theorem 1.1(ii) with cases (b) and (c) in Remark 1.2 below). Also, the nonlinearity u^p in Theorem 1.1 cannot be replaced with a slowly growing one. Namely, we show in Proposition 5.1 below that when $f(u) = u[\log(1+u)]^a$ with $1 < a < 2$, and for suitable potentials whose zeros satisfy (1.7), there exist solutions such that $u_t \ge 0$ and which blow up at every point of the domain. On the other hand, it is an interesting open problem what happens if assumption (1.7) is dropped, for instance if there is a line of zeros of V connecting x_0 to $\partial\Omega$. The question seems delicate, especially for $n \geq 2$.

Our next main result is a *local version* of a global result of Merle-Zaag [23]. It asserts that the solution of the subcritical nonlinear heat equation behaves like the corresponding ODE, in the sense that the diffusion term becomes asymptotically of smaller order than the reaction term wherever the solution is large. This result is crucial to our proof of Theorem 1.1.

Theorem 1.2. *Let $\omega \subset\subset D \subset\subset \Omega$. Assume* (1.2)-(1.6)*, $p < p_S$ and*

$$V(x) \geq c_0 \quad \text{in } D \tag{1.8}$$

for some $c_0 > 0$. Let u be a nonnegative classical solution of (1.1) *such that*

$$u(t,x) \leq M(T-t)^{-\alpha} \quad \text{in } (T/4,T) \times D \tag{1.9}$$

for some $M > 0$. Then for each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\big|u_t - V(x)f(u)\big| \leq \varepsilon u^p + C_\varepsilon \quad \text{in } [T/2,T) \times \omega. \tag{1.10}$$

Remark 1.1. *(a) Theorem 1.2 seems to be new even for $V \equiv 1$. In the result of* [23]*, it is assumed that u satisfies the Dirichlet boundary conditions on $\partial\Omega$, that Ω is convex, $D = \omega = \Omega$ and $V \equiv 1$. The case of nonconstant V is mentioned in* [23, p. 142]*, but (cf.* [34]*) it is implicitly assumed there that V is positive on $\overline{\Omega}$. On the other hand, in* [24]*, the same global result as in* [23] *is obtained for the Neumann problem, without convexity assumption on Ω.*

(b) The assumption $p < p_S$ is essentially optimal. Indeed, for $p_S < p < p_L := 1 + 6/(n-10)_+$ there exist self-similar, positive classical solutions of $u_t = \Delta u + u^p$, of the form $u(t,x) = (T-t)^{-\alpha}U(|x|/\sqrt{T-t})$ in $\mathbb{R}^n \times (0,T)$, with U bounded and $\Delta U(0) < 0$ (see [20, 2, 21, 25]*). In particular, these solutions satisfy* (1.9) *and*

$$-\Delta u(t,0) = \theta u^p(t,0) \to \infty, \qquad \text{as } t \to T,$$

for some $\theta \in (0,1)$, hence (1.10) *is violated.*

(c) Let D be an annulus, $D = \{x \in \mathbb{R}^n : \ r_1 < |x| < r_2\}$ with $r_2 > r_1 > 0$, and assume that u and V are radially symmetric. Then Theorem 1.2 remains true for all $p > 1$. Moreover, estimate (1.9) *is also true for all $p > 1$ (cf.* [27]*).*

The proof of Theorem 1.2 is done by rescaling arguments, relying on a Liouville type theorem of [23] for ancient solutions of $u_t - \Delta u = u^p$. In this sense it follows the scheme of proof of the corresponding global result in [23]. However, the proofs in both [23] and [24] make extensive use of the so-called weighted energy of Giga and Kohn [12], which requires working with prescribed boundary conditions. This tool is required in order to guarantee an upper type I estimate, as well as to avoid degeneracy of blowup. In the local case, we are here able to avoid any energy argument and to replace this ingredient by a different nondegeneracy property which is of purely local nature (see Proposition 2.1 and Lemma 3.2 below).[1]

As for the local type I estimate (cf. assumption (1.9) in Theorem 1.2), it is guaranteed by the following known result, which is a consequence of a different Liouville-type Theorem from [27] and [28] (see Theorem 3.1 and Remark 4.3(b) in [27]; note that the proof is given there for constant V, but it carries over with straightforward changes).

Theorem A. *Assume* (1.2)-(1.6)*,* $p < p_S$*,* $D \subset\subset \Omega$*,* (1.8) *for some* $c_0 > 0$*, and let* u *be a nonnegative classical solution of* (1.1)*. Assume in addition that one of the following conditions holds:*

$$\text{(A) } u_t \geq 0, \quad \text{(B) } n \leq 2, \quad \text{(C) } n \geq 3 \text{ and } p < n(n+2)/(n-1)^2.$$

Then there exists $M > 0$ *such that estimate* (1.9) *holds.*

Remark 1.2. *Let us summarize the previously known results about blowup or non blowup at zero points of the potential for positive solutions of the equation*

$$u_t = \Delta u + V(x)u^p, \quad 0 < t < T,\ x \in \Omega,$$

with $p > 1$ *and homogeneous Dirichlet boundary conditions (if* $\Omega \neq \mathbb{R}^n$*).*

(a) $\Omega = B_R$*,* $V(x) = |x|^\sigma$*,* $\sigma > 0$*,* $n \geq 3$*,* $p < 1 + 2\sigma/(n-1)$ *(with equality permitted if* $n = 3$*). If* u *is symmetric, then* 0 *is not a blowup point* [14]*;*

(b) $\Omega = B_R$*,* $V(x) = |x|^\sigma$*,* $\sigma > 0$*,* $n = 3$*,* $p > p_S + 2\sigma$*. Then there exist symmetric solutions such that* 0 *is a blowup point* [16]*;*

(c) $\Omega = \mathbb{R}^n$*,* $V(x) = |x|^\sigma$*,* $\sigma > 0$*,* $3 \leq n \leq 10+2\sigma$*,* $p > p_S + 2\sigma/(n-2)$ *(or an additional restriction for* $n > 10+2\sigma$*). Then there exist symmetric (backward self-similar) solutions such that* 0 *is a blowup point* [6]*;*

(d) Ω *bounded,* $0 \leq V$ *Hölder continuous in* $\overline{\Omega}$*. If* u *is nondecreasing in time, then blowup cannot occur at any zero point of* V [16]*.*

Moreover, in examples (b) and (d), the blowup is of type II. Finally, for the case of equation (1.1) *with* $f(0) > 0$*, specifically* $f(u) = (1+u)^p$ *or* $f(u) = e^u$*, blowup at zero points of* V *was excluded in* [15] *under the assumption that* u *is nondecreasing in time and the blowup set of* u *is a compact subset of* Ω*. However, for the Dirichlet problem,*

[1] Actually by the proof of Theorem 1.2, we see that the result remains true if V also depends on t (with $V(t,x)$ Hölder continuous satisfying (1.8) in $[T/2,T) \times D$), in which case no energy structure is available in general, even under prescribed boundary conditions.

the latter condition is only known to hold when Ω is convex and the potential $V(x)$ is monotonically decreasing near the boundary.

For results on other aspects of equation (1.1) with variable potential and power-like nonlinearity, we refer to, e.g., [26, 29, 22, 4, 3, 1, 32, 7]. For other applications of Merle-Zaag's Liouville theorem and ODE behavior in the study of the blowup set in the case $V \equiv 1$, see [33, 9] and the references therein.

Let us now explain the difficulties in ruling out the possibility of blowup at zero points of the potential V, and the new ideas to overcome these difficulties. It is already known [16, 15] that *type I* blowup cannot occur at a zero point of the potential. This follows from a local comparison with a suitable self-similar supersolution (recalled in Lemma 4.3 below). Consequently, we are reduced to proving an a priori type I estimate for the solution u near a blowup point x_0. A classical way to derive such an estimate is to apply the maximum principle to an auxiliary function of the form $J = u_t - \varepsilon f(u)$, so as to show $J \geq 0$ (cf. [8]). When $f(0) = 0$, for the Dirichlet problem, one can work on the whole domain Ω, taking advantage of the fact that $J = 0$ on $\partial\Omega$. When $f(0) > 0$ (or without prescribed boundary conditions), this is no longer possible. When the blowup set is compact (cf. Remark 1.2), this can be fixed by working on a subdomain of Ω (cf. [15]). For the corresponding quenching problem, with $f(u) = (1-u)^{-p}$, the possible lack of compactness of the blowup set, pointed as an obstacle in [18, 5], was overcome in [17] by introducing a modified functional $J = u_t - \varepsilon a(x)f(u)$, where $a(x)$ is a suitable function vanishing on the boundary. However this construction does not seem to work for the blowup problem, and we here use a completely different idea. Namely by using a Liouville type theorem of Merle and Zaag [23] (see Theorem B in section 3 below), we can show (cf. Theorem 1.2) that, near any point where V is nonzero and u is large enough, u_t is of same order as u^p, hence $J \geq 0$. If $V(x_0) = 0$ and the connected component of $\mathcal{V}_0$ containing x_0 does not intersect $\partial\Omega$, this allows one to apply the maximum principle to J on a suitable subdomain containing x_0, and on the boundary of which V is positive.

The outline of the paper is as follows. In Section 2, we prove a nondegeneracy result which is one of the ingredients of the proof of Theorem 1.2. Then Theorem 1.2 is proved in Section 3. Section 4 is devoted to the proof of Theorem 1.1. Finally, in Section 5, we consider the case of weak nonlinearities, for which we prove blowup at zero points of the potential, and also obtain additional information on the blowup behavior which stands in contrast with the case of power nonlinearities with a potential.

2. A nondegeneracy result

As an ingredient to the proof of Theorem 1.2, we will prove the following nondegeneracy property, valid for any $p > 1$. It extends a result of Giga and Kohn (see Theorem 2.1 in [13] and cf. also Theorem 25.3 in [30]). Namely, condition (2.2) below involves the sharp constant κ, instead of a small number $\varepsilon > 0$ in [13].

Proposition 2.1. *Let $p > 1$, $r, T, A > 0$, $x_0 \in \mathbb{R}^n$ and u be a positive classical solution of the inequality*

$$u_t - \Delta u \le Au^p \quad \text{in } (0,T) \times B(x_0, r). \tag{2.1}$$

Assume that there exist $k \in (0, \kappa)$ and $\delta \in (0,T)$ such that

$$(T-t)^\alpha u(t,x) \le kA^{-\alpha} \quad \text{in } [T-\delta, T) \times B(x_0, r). \tag{2.2}$$

Then there exists a constant $L = L(p, n, A, k, r, \delta) > 0$ such that

$$u(t,x) \le L \quad \text{in } (T-\delta, T) \times B(x_0, r/4).$$

Proof. Assume $x_0 = 0$ without loss of generality. For any $\sigma \in (0,2)$ and $R \in (0,r]$, we may find $\phi_R \in C^2(\mathbb{R}^n)$ such that $0 \le \phi_R \le 1$,

$$\phi_R(x) = 0 \ \text{ for } |x| \ge 2R/3, \qquad \phi_R(x) = 1 \ \text{ for } |x| \le R/2, \tag{2.3}$$

and

$$|\nabla\phi_R|^2 + |\Delta\phi_R^2| \le C(R,n)\phi_R^\sigma. \tag{2.4}$$

Indeed, this function can be constructed as follows. We fix a nonincreasing function $h \in C^3([1/2, a])$ with $a = 2/3$, such that $h(1/2) = 1$, $h'(1/2) = h''(1/2) = 0$, $h(a) = h'(a) = h''(a) = 0$ and $h'''(a) = -6$. We then let

$$\psi(s) = \begin{cases} 1, & s \le 1/2 \\ (h(s))^l, & 1/2 < s < a \\ 0, & s \ge a, \end{cases}$$

with $l \ge 2$ an integer. We note that $h''(s) \sim 6(a-s)$, $h'(s) \sim -3(a-s)^2$ and $h(s) \sim (a-s)^3$, as $s \to a_-$, so that

$$\psi(s) \sim (a-s)^{3l}, \quad \psi'(s) \sim c_1(a-s)^{3l-1}, \quad \psi''(s) \sim c_2(a-s)^{3l-2}, \tag{2.5}$$

as $s \to a_-$. Finally setting $\phi_R(x) = \psi(|x|/R)$, we see that ϕ_R has the desired properties (in particular, (2.4) follows from (2.5) if l is large enough).

Let $\varepsilon > 0$ and put

$$v = v_{\varepsilon,R} = u^{1+\varepsilon}\phi^2, \quad \text{with } \phi = \phi_R.$$

For $(t,x)\in(T-\delta,T)\times B_R$, we have

$$\begin{aligned}v_t-\Delta v&=(1+\varepsilon)u^\varepsilon u_t\phi^2-(1+\varepsilon)\phi^2(u^\varepsilon\Delta u+\varepsilon u^{\varepsilon-1}|\nabla u|^2)\\&\quad-4(1+\varepsilon)u^\varepsilon\phi\nabla u\cdot\nabla\phi-u^{1+\varepsilon}\Delta\phi^2.\end{aligned}$$

Using (2.1) and $4|u^\varepsilon\phi\nabla u\cdot\nabla\phi|\le\varepsilon u^{\varepsilon-1}\phi^2|\nabla u|^2+4\varepsilon^{-1}u^{1+\varepsilon}|\nabla\phi|^2$, we obtain

$$v_t-\Delta v\le(1+\varepsilon)Au^{p+\varepsilon}\phi^2+(1+\varepsilon)u^{1+\varepsilon}(4\varepsilon^{-1}|\nabla\phi|^2+|\Delta\phi^2|).$$

Moreover, by (2.4) with $\sigma=2(1+\varepsilon)/(p+\varepsilon)$ and Young's inequality, we get

$$\begin{aligned}&(1+\varepsilon)u^{1+\varepsilon}(4\varepsilon^{-1}|\nabla\phi|^2+|\Delta\phi^2|)\\&\qquad=(1+\varepsilon)u^{1+\varepsilon}\phi^{2(1+\varepsilon)/(p+\varepsilon)}\phi^{-2(1+\varepsilon)/(p+\varepsilon)}(4\varepsilon^{-1}|\nabla\phi|^2+|\Delta\phi^2|)\\&\qquad\le\varepsilon Au^{p+\varepsilon}\phi^2+B\end{aligned}$$

where $B=B(\varepsilon,A,p,n,r)>0$ (the computation is valid at any point such that $\phi>0$, but the conclusion is also true where $\phi=0$, by (2.4)). It follows that

$$v_t-\Delta v\le(1+2\varepsilon)Au^{p-1}v+B\quad\text{in }(T-\delta,T)\times B_R.\tag{2.6}$$

Set $m:=(1+2\varepsilon)k^{p-1}$. Since $k<\kappa=\alpha^\alpha$, we may choose $\varepsilon=\varepsilon(p,k)>0$ sufficiently small, so that $m<\alpha$. We first use the choice $R=r$. Using (2.1) and assumption (2.2), we see that $v=v_{\varepsilon,r}$ satisfies

$$v_t-\Delta v\le m(T-t)^{-1}v+B\quad\text{in }(T-\delta,T)\times B_r.$$

For $K>0$, setting

$$\overline{v}=\overline{v}(t):=K(T-t)^{-m}-B(m+1)^{-1}(T-t),\quad T-\delta\le t<T,$$

we have

$$\overline{v}_t=Km(T-t)^{-m-1}+B(m+1)^{-1}=m(T-t)^{-1}\overline{v}+B\quad\text{in }(T-\delta,T).$$

Moreover, taking

$$K\ge K(\delta,A,p,n,k,r):=B(m+1)\delta^{m+1}+(kA^{-\alpha})^{1+\varepsilon}\delta^{m-\alpha(1+\varepsilon)},$$

it follows from (2.2) that

$$\overline{v}(T-\delta)=K\delta^{-m}-B(m+1)^{-1}\delta\ge\left[k(A\delta)^{-\alpha}\right]^{1+\varepsilon}\ge v(T-\delta,x),\quad x\in B_r.$$

Since $v=0$ on $(T-\delta,T)\times\partial B_r$, we then deduce from the comparison principle that $v\le\overline{v}$ in $[T-\delta,T)\times B_r$, hence

$$u^{1+\varepsilon}\le K(T-t)^{-m}\quad\text{in }(T-\delta,T)\times B_{r/2}.\tag{2.7}$$

We next use the choice $R=r/2$, with ε as above. Going back to (2.6) and using (2.7), we see that $v=v_{\varepsilon,r/2}$ satisfies

$$v_t-\Delta v\le K_1(T-t)^{-\gamma}v+\tilde{B}\quad\text{in }(T-\delta,T)\times B_{r/2},$$

with

$$\gamma := \frac{(p-1)m}{1+\varepsilon} < 1,\ K_1 := (1+2\varepsilon)AK^{(p-1)/(1+\varepsilon)},\ \tilde{B} = \tilde{B}(\varepsilon, A, p, n, r) > 0.$$

Setting $K_2 = 2K_1/(1-\gamma)$ and

$$\overline{w} = \overline{w}(t) := K_3 \exp[-K_2(T-t)^{1-\gamma}], \quad T-\delta \le t < T,$$

with $K_3 = K_3(\delta, A, p, n, k, r) > 0$ sufficiently large, we see that

$$\overline{w}_t = 2K_1(T-t)^{-\gamma}\overline{w} \ge K_1(T-t)^{-\gamma}\overline{w} + \tilde{B} \quad \text{in } (T-\delta, T),$$

as well as $\overline{w}(T-\delta) \ge v(T-\delta, x)$ in $B_{r/2}$. We then deduce from the comparison principle that $v \le \overline{w}$ in $[T-\delta, T) \times B_{r/2}$, hence

$$u^{1+\varepsilon} \le K_3 \quad \text{in } (T-\delta, T) \times B_{r/4}.$$

This concludes the proof. □

3. Proof of Theorem 1.2

The proof of Theorem 1.2 relies on the following Liouville type result, which is due to Merle and Zaag [23].

Theorem B. *Let $1 < p < p_S$, let $T \ge 0$ and let z be a nonnegative classical solution of*

$$z_t - \Delta z = z^p \quad \text{in } (-\infty, T) \times \mathbb{R}^n,$$

such that

$$\sup_{(t,x)\in(-\infty,T)\times\mathbb{R}^n} (T-t)^\alpha z(t,x) < \infty.$$

Then z is independent of x.

For the proof of Theorem 1.2, we need the following two lemmas. They are consequences of Theorem B and of Proposition 2.1, respectively.

Lemma 3.1. *Assume* (1.2)-(1.6) *and*

$$V(x) \ge c_0 \quad \text{in } D \tag{3.1}$$

for some $c_0 > 0$ and let $\omega \subset\subset D \subset\subset \Omega$. Let u be a nonnegative classical solution u of (1.1).

(i) *Assume that $p < p_S$ and that u satisfies*

$$u(t,x) \le M(T-t)^{-\alpha} \quad \text{in } (T/4, T) \times D. \tag{3.2}$$

Then for each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|\Delta u(t,x)| \le \varepsilon(T-t)^{-\alpha-1} + C_\varepsilon \quad \text{in } (T/2, T) \times \omega \tag{3.3}$$

and

$$|\nabla u(t,x)| \le \varepsilon(T-t)^{-\alpha-(1/2)} + C_\varepsilon \quad \text{in } (T/2, T) \times \omega. \tag{3.4}$$

(ii) *Let* $p > 1$, *let* D *be an annulus,* $D = \{x \in \mathbb{R}^n : \ r_1 < |x| < r_2\}$ *with* $r_2 > r_1 > 0$, *and assume that* u *and* V *are radially symmetric. Then* (3.3) *and* (3.4) *are true.*

Lemma 3.2. *Let* $p > 1$, $r, T, A > 0$, $x_0 \in \mathbb{R}^n$ *and* u *be a positive classical solution of the inequality*

$$u_t - \Delta u \le A u^p \quad \text{in } (0,T) \times B(x_0, r). \tag{3.5}$$

Assume that, for each $\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$ *such that*

$$\Delta u(t,x) \le \varepsilon (T-t)^{-\alpha-1} + C_\varepsilon \quad \text{in } (T/2, T) \times B(x_0, r). \tag{3.6}$$

(i) *For any* $k \in (0, \kappa)$, *there exist constants* $\tau_0, K > 0$ *with the following property: If*

$$(T-t_0)^\alpha u(t_0, \cdot) \le k A^{-\alpha} \quad \text{in } B(x_0, r), \quad \text{for some } t_0 \in [(T-\tau_0)_+, T). \tag{3.7}$$

then

$$u(t,x) \le K \quad \text{in } [t_0, T) \times B(x_0, r/4).$$

Here τ_0 *depends only on* p, k, A *and on the constants* C_ε, *and* K *depends only on* p, k, A, n, r, t_0 *and on the constants* C_ε.

(ii) *If* x_0 *is a blowup point of* u *(i.e.,* $\limsup_{t \to T,\, x \to x_0} u(t,x) = \infty$*), then*

$$\liminf_{t \to T} (T-t)^\alpha u(t, x_0) \ge \kappa A^{-\alpha}. \tag{3.8}$$

Proof of Lemma 3.1. (i) Assume the contrary. Then there exist $\varepsilon > 0$ and a sequence of points $(t_j, x_j) \in (T/2, T) \times \omega$ such that

$$|\Delta u(t_j, x_j)| + |\nabla u(t_j, x_j)|^{2(\alpha+1)/(2\alpha+1)} \ge \varepsilon (T-t_j)^{-\alpha-1} + j. \tag{3.9}$$

We may assume that $x_j \to x_\infty \in \overline{\omega}$. We also have $t_j \to T$, since otherwise, $\Delta u(t_j, x_j)$ and $\nabla u(t_j, x_j)$ would be bounded. We rescale u by setting:

$$v_j(s,y) := \lambda_j^{2\alpha} u(t_j + \lambda_j^2 s, x_j + \lambda_j y), \quad \text{in } D_j := \{-t_j \lambda_j^{-2} < s < 1, \ |y| < d\lambda_j^{-1}\},$$

where $\lambda_j = \sqrt{T - t_j} \to 0$ and $d = \mathrm{dist}(\omega, D^c) > 0$. By a simple computation, we see that

$$\partial_s v_j - \Delta_y v_j = V_j(y) f_j(v_j(s,y)) \quad \text{in } D_j, \tag{3.10}$$

where $f_j(v) := \lambda_j^{2p/(p-1)} f(\lambda_j^{-2/(p-1)} v)$ and $V_j(y) := V(x_j + \lambda_j y)$.

As a consequence of assumption (3.2), we note that

$$v_j(s,y) \le M \lambda_j^{2\alpha} (T - t_j - \lambda_j^2 s)^{-\alpha} = M(1-s)^{-\alpha} \quad \text{in } D_j. \tag{3.11}$$

Moreover, we have

$$\begin{aligned}
|\Delta_y v_j(0,0)| &+ |\nabla_y v_j(0,0)|^{2(\alpha+1)/(2\alpha+1)} \\
&= \lambda_j^{2\alpha+2} \Big(|\Delta u(t_j, x_j)| + |\nabla u(t_j, x_j)|^{2(\alpha+1)/(2\alpha+1)} \Big) \\
&= (T-t_j)^{\alpha+1} \Big(|\Delta u(t_j, x_j)| + |\nabla u(t_j, x_j)|^{2(\alpha+1)/(2\alpha+1)} \Big)
\end{aligned}$$

hence, by (3.9),

$$|\Delta_y v_j(0,0)| + |\nabla_y v_j(0,0)|^{2(\alpha+1)/(2\alpha+1)} \geq \varepsilon. \tag{3.12}$$

Note that $f_j(0) = \lambda_j^{2p/(p-1)} f(0) \leq 1$ and that, by (1.5),

$$|f_j'(v)| = \lambda_j^2 |f'(\lambda_j^{-2/(p-1)} v)| \leq C\lambda_j^2 \big[1 + (\lambda_j^{-2/(p-1)} v)^{p-1}\big] \leq C(1 + v^{p-1}).$$

As a consequence of (3.10), (3.11), (1.6) and interior parabolic estimates, we then deduce that there exist a function $w \geq 0$ and $\nu \in (0,1)$ such that, for each compact subset K of $E = (-\infty, 1) \times \mathbb{R}^n$, the sequence v_j converges in $C^{1+(\nu/2),2+\nu}(K)$ to w as $j \to \infty$. In particular, by (3.11) and (3.12), we have

$$w(s,y) \leq M(1-s)^{-\alpha} \quad \text{in } E, \tag{3.13}$$

as well as

$$|\Delta_y w(0,0)| + |\nabla_y w(0,0)|^{2(\alpha+1)/(2\alpha+1)} \geq \varepsilon. \tag{3.14}$$

Now, for each $(s,y) \in E$, considering separately the cases $w(s,y) > 0$ and $w(s,y) = 0$ and using assumption (1.4), we see that

$$\lim_j \lambda_j^{2p/(p-1)} f(\lambda_j^{-2/(p-1)} v(s,y)) = w^p(s,y).$$

It follows that w is a classical solution of

$$\partial_s w - \Delta_y w = V(x_\infty) w^p \quad \text{in } E. \tag{3.15}$$

Note that $V(x_\infty) > 0$ by (3.1). By Theorem B, such a solution with the additional property (3.13) must necessarily be spatially homogeneous. This contradicts (3.14).

(ii) We only sketch the necessary changes. Since u is radial, setting $\rho = |x|$, $\rho_j = |x_j|$ and $\rho = \rho_j + \lambda_j y$, equation (3.10) can be written as

$$\partial_s v_j - \partial_y^2 v_j = V_j(y) f_j(v_j(s,y)) + \frac{n-1}{\rho_j + \lambda_j y} \lambda_j^{2\alpha+2} u_\rho(t_j + \lambda_j^2 s, \rho_j + \lambda_j y).$$

On the other hand, in the radial case in an annulus, by [27], estimate (3.2) is true and moreover

$$|u_\rho(t,\rho)| \leq M_1 (T-t)^{-\alpha-(1/2)} \quad \text{in } (T/4, T) \times (r_1, r_2),$$

for some $M_1 > 0$. Therefore, since $\lambda_j = \sqrt{T - t_j}$, we have

$$\begin{aligned} \frac{n-1}{\rho_j + \lambda_j y} \lambda_j^{2\alpha+2} |u_\rho(t_j + \lambda_j^2 s, \rho_j + \lambda_j y)| &\leq \frac{n-1}{r_1} (T-t_j)^{\alpha+1} M_1 (T - t_j - \lambda_j^2 s)^{-\alpha-(1/2)} \\ &\leq \frac{n-1}{r_1} (T-t_j)^{1/2} M_1 (1-s)^{-\alpha-(1/2)} \end{aligned}$$

in D_j. Consequently, the limiting equation (3.15) becomes

$$\partial_s w - w_{yy} = V(x_\infty) w^p \quad \text{in } (-\infty, 1) \times \mathbb{R},$$

so that we can conclude as before. $\square$

Proof of Lemma 3.2. (i) Let $\varepsilon > 0$. By our assumptions, we have

$$u_t \leq Au^p + \varepsilon(T-t)^{-\alpha-1} + C_\varepsilon \quad \text{in } (T/2, T) \times B(x_0, r).$$

Since $k \in (0, \kappa)$, we may choose B so that

$$kA^{-\alpha} < B < \kappa A^{-\alpha}. \tag{3.16}$$

Set

$$\phi(t) := B(T-t)^{-\alpha}.$$

We claim that there exist $\varepsilon = \varepsilon(p, k, A) > 0$ and $\tau_0 = \tau_0(p, k, A, C_\varepsilon) > 0$ small, such that

$$\phi'(t) \geq A\phi^p + \varepsilon(T-t)^{-\alpha-1} + C_\varepsilon \quad \text{for all } t \in [(T-\tau_0)_+, T). \tag{3.17}$$

Indeed, since $p\alpha = \alpha + 1$, (3.17) is equivalent to

$$B\alpha - AB^p - \varepsilon \geq C_\varepsilon(T-t)^{p\alpha} \quad \text{for all } t \in [(T-\tau_0)_+, T).$$

Since $B\alpha - AB^p > 0$ by (3.16), we see that this is true if we choose $\varepsilon > 0$ small and then τ_0 small, with the dependence specified above.

Now (3.16) and assumption (3.7) guarantee that, for each $x \in B(x_0, r)$, we have $u(t_0, x) \leq \phi(t_0)$. By (3.17) and ODE comparison, it follows that $u(t, x) \leq \phi(t)$ for all $t \in [t_0, T)$. Consequently, we have

$$u(t, x) \leq B(T-t)^{-\alpha} \quad \text{in } [t_0, T) \times B(x_0, r).$$

Since $B < \kappa A^{-\alpha}$, Proposition 2.1 then guarantees the desired bound.

(ii) If (3.8) fails, then there exist $k < \kappa$ and $t_0 \in ((T-\tau_0)_+, T)$ such that $(T - t_0)^\alpha u(t_0, x_0) < kA^{-\alpha}$, where τ_0 is given by assertion (i). By continuity there exists $r > 0$ such that $(T-t_0)^\alpha u(t_0, \cdot) < kA^{-\alpha}$ in $B(x_0, r)$. By assertion (i), it follows that x_0 is not a blowup point. $\square$

We are now in a position to give the proof of Theorem 1.2, by combining Lemmas 3.1-3.2 and an appropriate rescaling argument.

Proof of Theorem 1.2. Assume for contradiction that there exist $c_1 > 0$ and a sequence $(t_j, x_j) \in [T/2, T) \times \omega$ such that

$$|\Delta u(t_j, x_j)| \geq c_1 u^p(t_j, x_j) + j. \tag{3.18}$$

Step 1. *Nondegeneracy at points x_j.* First, it follows from (3.3) in Lemma 3.1(i) and (3.18) that, for all $\varepsilon > 0$,

$$c_1 u^p(t_j, x_j) + j \leq |\Delta u(t_j, x_j)| \leq \varepsilon(T-t_j)^{-\alpha-1} + C_\varepsilon.$$

Therefore, $t_j \to T$ and

$$u(t_j, x_j) \leq \varepsilon_j (T-t_j)^{-\alpha} \tag{3.19}$$

with $\varepsilon_j \to 0$ as $j \to \infty$. Moreover, we may also assume that $x_j \to x_\infty \in \overline{\omega}$. Note that x_∞ is in particular a blowup point (i.e., $\limsup_{t\to T,\, x\to x_\infty} u(t,x) = \infty$), since otherwise by parabolic regularity, $\Delta u(t_j, x_j)$ would be bounded.

We claim that there exists a subsequence of $\{(t_j, x_j)\}$ (not relabeled) and a sequence $\hat{t}_j \to T$ such that

$$\hat{t}_j \in (0, t_j) \quad \text{and} \quad (T - \hat{t}_j)^\alpha u(\hat{t}_j, x_j) = \frac{\kappa}{2} V^{-\alpha}(x_\infty). \tag{3.20}$$

To prove the claim, in view of (3.19), by continuity, it suffices to show that, for each $\eta > 0$ and each $j_0 \geq 1$ there exist $j \geq j_0$ and $t \in (T - \eta, t_j)$ such that

$$(T - t)^\alpha u(t, x_j) \geq \frac{\kappa}{2} V^{-\alpha}(x_\infty).$$

If this were false, then there would exist $\eta > 0$ and $j_0 \geq 1$ such that, for all $j \geq j_0$ and $t \in (T-\eta, t_j)$, $(T-t)^\alpha u(t, x_j) < \frac{\kappa}{2} V^{-\alpha}(x_\infty)$. For each given $t \in (T-\eta, T)$, since $t_j \to T$, we would have $t_j > t$ for j sufficiently large, hence $(T - t)^\alpha u(t, x_\infty) \leq \frac{\kappa}{2} V^{-\alpha}(x_\infty)$, by letting $j \to \infty$. Using the continuity of V and applying Lemma 3.2(ii) for some $A > V(x_\infty)$ close to $V(x_\infty)$, we would deduce that x_∞ is not a blowup point, which is a contradiction. This proves the claim.

Step 2. *Rescaling and convergence to a bounded flat profile.* We rescale similarly as in the proof of Lemma 3.1, but now taking $\hat{t}_j$ as rescaling times. Namely, we set:

$$v_j(s, y) := \lambda_j^{2\alpha} u(\hat{t}_j + \lambda_j^2 s, x_j + \lambda_j y) \quad \text{in } \hat{D}_j := \{-\hat{t}_j \lambda_j^{-2} < s < 1,\ |y| < d\lambda_j^{-1}\},$$

where $\lambda_j = \sqrt{T - \hat{t}_j} \to 0$ and $d = \mathrm{dist}(\omega, D^c) > 0$. We have

$$\partial_s v_j - \Delta_y v_j = V(x_j + \lambda_j y) \lambda_j^{2p/(p-1)} f(\lambda_j^{-2/(p-1)} v_j(s, y)) \quad \text{in } \hat{D}_j. \tag{3.21}$$

As a consequence of assumption (1.9), we note that

$$v_j(s, y) \leq M \lambda_j^{2\alpha} (T - \hat{t}_j - \lambda_j^2 s)^{-\alpha} = M(1 - s)^{-\alpha} \quad \text{in } \hat{D}_j. \tag{3.22}$$

By (3.3)-(3.4), for all $\varepsilon > 0$, we have

$$\begin{aligned} |\nabla_y v_j(s, y)| &= \lambda_j^{2\alpha+1} |\nabla u(\hat{t}_j + \lambda_j^2 s, x_j + \lambda_j y)| \\ &\leq \lambda_j^{2\alpha+1} \left[\varepsilon(T - \hat{t}_j - \lambda_j^2 s)^{-\alpha-(1/2)} + C_\varepsilon\right] \\ &= \varepsilon(1 - s)^{-\alpha-(1/2)} + \lambda_j^{2\alpha+1} C_\varepsilon \quad \text{in } \hat{D}_j \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} |\Delta_y v_j(s, y)| &= \lambda_j^{2\alpha+2} |\Delta u(\hat{t}_j + \lambda_j^2 s, x_j + \lambda_j y)| \\ &\leq \lambda_j^{2\alpha+2} \left[\varepsilon(T - \hat{t}_j - \lambda_j^2 s)^{-\alpha-1} + C_\varepsilon\right] \\ &= \varepsilon(1 - s)^{-\alpha-1} + \lambda_j^{2\alpha+2} C_\varepsilon \quad \text{in } \hat{D}_j. \end{aligned} \tag{3.24}$$

Also, letting

$$s_j := \frac{t_j - \hat{t}_j}{T - \hat{t}_j} \in (0, 1),$$

by (3.18), we see that

$$|\Delta_y v_j(s_j,0)| = \lambda_j^{2\alpha+2}|\Delta u(t_j,x_j)| \geq c_1(T-\hat{t}_j)^{p\alpha}u^p(t_j,x_j) = c_1 v_j^p(s_j,0). \tag{3.25}$$

Moreover, by (3.20), we have

$$v_j(0,0) = \lambda_j^{2\alpha}u(\hat{t}_j,x_j) = \frac{\kappa}{2}V^{-\alpha}(x_\infty). \tag{3.26}$$

As a consequence of (3.23) and (3.26), it follows that

$$v_j(0,y) \to \frac{\kappa}{2}V^{-\alpha}(x_\infty), \text{ as } j\to\infty, \text{ uniformly for } y \text{ in compact subsets of } \mathbb{R}^n. \tag{3.27}$$

By (3.21), (3.22), (1.6) and interior parabolic estimates, we then deduce that there exist a function $w \geq 0$ and $\nu \in (0,1)$ such that, for each compact subset K of $[0,1)\times\mathbb{R}^n$, the sequence v_j converges in $C^{1+(\nu/2),2+\nu}(K)$ to w as $j\to\infty$. By (3.21) and (3.24), the function w solves

$$\partial_s w(s,y) = V(x_\infty)w^p(s,y), \quad 0 \leq s < 1,\ y \in \mathbb{R}^n$$

with

$$w(0,y) = \frac{\kappa}{2}V^{-\alpha}(x_\infty).$$

Integrating this ODE, we obtain

$$w(s,y) = V^{-\alpha}(x_\infty)\kappa(2^{p-1}-s)^{-\alpha}, \quad 0 \leq s < 1,\ y \in \mathbb{R}^n. \tag{3.28}$$

Step 3. *Uniform regularity and flatness of rescaled solution and conclusion.* We shall now apply Lemma 3.2(i). First, since $w(s,y) \leq K_0 := V^{-\alpha}(x_\infty)\kappa(2^{p-1}-1)^{-\alpha}$, we deduce from Step 2 that for any $s_0 \in (0,1)$, there exists $j_1 \geq 1$ such that

$$v_j(s_0,y) \leq K_0 + 1, \quad |y| \leq 2,\ j \geq j_1. \tag{3.29}$$

By assumption (1.4), there exists $C > 0$ such that $f(u) \leq \frac{3}{2}(u+C)^p$ for all $u \geq 0$. This along with (3.21) and the continuity of V implies that $z_j := v_j + 1$ satisfies

$$\partial_s z_j - \Delta_y z_j = \partial_s v_j - \Delta_y v_j \leq 2V(x_\infty)\lambda_j^{2p/(p-1)}(\lambda_j^{-2/(p-1)}v_j + C)^p \quad \text{ in } \hat{D}_j,$$

hence

$$\partial_s z_j - \Delta_y z_j \leq 2V(x_\infty)z_j^p, \quad 0 \leq s < 1,\ |y| \leq 2, \tag{3.30}$$

for all j sufficiently large. Let τ_0 be given by Lemma 3.2(i) with $A = 2V(x_\infty)$, $r = 2$, $T = 1$, $k = \kappa/2$. Choosing $s_0 \in [(1-\tau_0)_+, 1)$ such that

$$(1-s_0)^\alpha(K_0+2) < \frac{\kappa}{2}(2V(x_\infty))^{-\alpha},$$

in view of (3.29) and (3.30), it then follows from Lemma 3.2(i) that

$$v_j(s,y) \leq z_j(s,y) \leq K, \quad s_0 \leq s < 1,\ |y| \leq 1/2,\ j \geq j_1.$$

Going back to equation (3.21), and using parabolic estimates, we deduce that v_j actually converges to w in $C^{1+(\nu/2),2+\nu}([1/2,1)\times B_{1/4})$. In view of (3.28), this implies

$$\liminf_{j\to\infty} v_j(s_j,0)\geq \frac{\kappa}{2}V^{-\alpha}(x_\infty)\quad\text{and}\quad \lim_{j\to\infty}\Delta v_j(s_j,0)=0.$$

This contradicts (3.25) and the proof is completed.

Finally, we note that in the radial case in an annulus (cf. Remark 1.1(c)), the above proof remains valid for all $p>1$, using assertion (ii) of Lemma 3.1. $\square$

4. Proof of Theorem 1.1

The proof is carried out through a series of lemmas. In what follows we denote $\{V\leq k\}=\{x\in\overline{\Omega}:\ V(x)\leq k\}$. We begin with the following simple topological lemma.

Lemma 4.1. *Let Ω be a bounded domain of $\mathbb{R}^n$ and let $V:\overline{\Omega}\to[0,\infty)$ be a continuous function. Let $x_0\in\Omega$ and denote by A_0 the connected component of $\mathcal{V}_0$ containing x_0. If $A_0\cap\partial\Omega=\emptyset$, then there exists a subdomain $\Omega_0\subset\subset\Omega$ and $\eta>0$ such that $x_0\in\Omega_0$ and*

$$V\geq\eta>0\quad \textit{on } \partial\Omega_0.$$

Proof. For any positive integer m, we denote by A_m the connected component of $\{V\leq 1/m\}$ containing x_0.

Step 1. We claim that we have

$$A_0\subset A_p\subset A_m,\ \text{for all } p\geq m\geq 1,\tag{4.1}$$

$$A_m \text{ is closed,}\tag{4.2}$$

$$A_0=\bigcap_{m\geq 1}A_m.\tag{4.3}$$

Assertion (4.1) follows directly from $\mathcal{V}_0\subset\{V\leq 1/p\}\subset\{V\leq 1/m\}$.

To show (4.2), first note that $\{V\leq 1/m\}$ is closed, due to the continuity of V, hence $\overline{A_m}\subset\{V\leq 1/m\}$. Also, since the closure of a connected set is connected, $\overline{A_m}$ is connected. Now, since $A_m\subset\overline{A_m}$ and, by definition, A_m is the largest connected subset of $\{V\leq 1/m\}$ containing x_0, we necessarily have $\overline{A_m}=A_m$, hence (4.2).

Let us prove (4.3). By (4.1), we have $A_0\subset K:=\bigcap_{m\geq 1}A_m$. By (4.2) and the boundedness of Ω, each A_m is compact. As a decreasing intersection of compact connected sets, K is thus connected (and compact). Moreover, we have $K\subset\bigcap_{m\geq 1}\{V\leq 1/m\}=\mathcal{V}_0$. Then, since $A_0\subset K$ and, by definition, A_0 is the largest connected subset of $\mathcal{V}_0$ containing x_0, we necessarily have $K=A_0$, hence (4.3).

Step 2. Next we claim that there exists $m\geq 1$ such that $A_m\cap\partial\Omega=\emptyset$. Indeed, otherwise, for each $m\geq 1$, we may find $x_m\in A_m\cap\partial\Omega$. Since $\partial\Omega$ is compact, up to a subsequence, we may assume $x_m \to y$ for some $y \in \partial\Omega$. For each $m \geq 1$, by (4.1), we have $x_p \in A_p \subset A_m$ for all $p \geq m$, hence $y \in A_m$ owing to (4.2). Therefore $y \in \bigcap_{m\geq 1} A_m = A_0$ in view of (4.3). But this is a contradiction with $A_0 \cap \partial\Omega = \emptyset$.

Now let A'_m denote the connected component of $\{V < 1/m\}$ containing x_0. Then $A'_m \subset A_m$ due to $\{V < 1/m\} \subset \{V \leq 1/m\}$. Since $A_m \cap \partial\Omega = \emptyset$ and A_m is compact, we have

$$A'_m \subset\subset \Omega. \tag{4.4}$$

We then claim that A'_m is open. Let $x \in A'_m \subset \Omega \cap \{V < 1/m\}$. Since V is continuous, there exists $\rho > 0$ such that $B_\rho(x) \subset \{V < 1/m\}$. Therefore, $A'_m \cup B_\rho(x) \subset \{V < 1/m\}$ is connected (union of two non-disjoint connected sets) and contains x_0. By the definition of A'_m we deduce that $A'_m \cup B_\rho(x) = A'_m$, hence $B_\rho(x) \subset A'_m$ and A'_m is open.

Finally, we observe that $V = 1/m > 0$ on $\partial A'_m$. Therefore, recalling (4.4), we see that $\Omega_0 = A'_m$ has all the required properties. □

As a consequence of Theorem 1.2 and Lemma 4.1, we next prove the following local type I blowup lemma.

Lemma 4.2. *Let the assumptions of Theorem 1.1 be in force. If x_0 is a blowup point, then blowup is of type I near x_0. More precisely, there exist $M, \rho > 0$ such that*

$$u(t,x) \leq M(T-t)^{-\alpha} \quad \text{in } (0,T) \times B(x_0,\rho). \tag{4.5}$$

Proof. Following [8], we set

$$J = u_t - \varepsilon f(u).$$

A standard computation yields, using that f is of class C^2 and convex,

$$\begin{aligned} J_t - \Delta J &= (u_t - \Delta u)_t - \varepsilon f'(u)(u_t - \Delta u) - \varepsilon f''(u)|\nabla u|^2 \\ &\leq V(x)f'(u)u_t - \varepsilon V(x)f'(u)f(u) = V(x)f'(u)J \quad \text{ in } (0,T)\times\Omega. \end{aligned} \tag{4.6}$$

As a consequence of Lemma 4.1, there exist a subdomain $\Omega_0 \subset\subset \Omega$ and $\eta > 0$ such that $x_0 \in \Omega_0$ and

$$V \geq 3\eta > 0 \quad \text{on } \partial\Omega_0. \tag{4.7}$$

Recall that $u_t \geq 0$. Since u blows up, we have $u_t \not\equiv 0$. Therefore, by the strong maximum principle applied to u_t, there exists $t_1 \in [T/2, T)$ such that

$$\gamma = \inf_{[t_1,T)\times\overline{\Omega}_0} u_t > 0. \tag{4.8}$$

In particular, assuming $\varepsilon \leq \varepsilon_0 := \gamma\Big(\sup_{x\in\overline{\Omega}_0} f(u(t_1,x))\Big)^{-1}$, we have

$$J(t_1,\cdot) \geq 0 \quad \text{ in } \Omega_0.$$

As a consequence of Theorem A, Theorem 1.2 and (4.7), along with the continuity of V, there exists $C > 0$ such that

$$u_t \geq 2\eta f(u) - C \quad \text{ in } (0,T) \times \partial\Omega_0. \tag{4.9}$$

(In the radial case, cf. assertion (ii) of Theorem 1.1, we use Remark 1.1(c).) Assume $\varepsilon < \min(\varepsilon_0, \eta, \gamma\eta/C)$. Combining (4.8) and (4.9), we obtain

$$J = u_t - \varepsilon f(u) \geq \max\Big[\eta f(u) - C,\ \gamma - \varepsilon f(u)\Big] \geq 0$$

in $(0,T)\times\partial\Omega_0$. We can then apply the maximum principle to deduce $J \geq 0$ in $[t_1,T)\times\Omega_0$. Using (1.4), by integration, estimate (4.5) follows on $[t_1,T)$, hence on $(0,T)$. □

Blowup at x_0 will be finally ruled out by the following lemma, which shows that type I blowup cannot occur at a zero point of the potential.

Lemma 4.3. *Let $p > 1$, $C > 0$ and $V \geq 0$ be a continuous function on $B(x_0,\rho)$ for some $x_0 \in \mathbb{R}^n$ and $\rho > 0$. Let $u \geq 0$ be a classical solution of*

$$u_t \leq \Delta u + C\,V(x)(1+u)^p \quad \text{in } (0,T) \times B(x_0,\rho)$$

and assume that u satisfies the type I estimate (4.5). If $V(x_0) = 0$, then x_0 is not a blowup point, i.e., there exists $r \in (0,\rho)$ such that u is bounded on $(T/2,T) \times B(x_0,r)$.

This was proved in [16]. For completeness, we reproduce the (supersolution) argument of [16] as follows.

Proof of Lemma 4.3. Let ρ, M be the constants in (4.5). Following [16], we introduce the function

$$w(t,x) = \frac{K}{[q(x)+(T-t)]^\alpha}, \quad q(x) = \beta\cos^2\left(\frac{\pi|x-x_0|}{2r}\right),$$

where $r \in (0,\rho/2)$ and the constants $\beta \in (0,1)$ and $K > M$ are to be determined later. Due to (4.5), we have

$$u(t,x) \leq w(t,x) \ \text{ for } (t,x) \in (0,T)\times\partial B_0, \quad B_0 := \{x \mid |x-x_0| < r\}.$$

Clearly, $u(0,x) \leq w(0,x)$ for $x \in \bar{B}_0$, if K is chosen sufficiently large. We set $f(w) = C(1+w)^p$ and compute

$$w_t - \Delta w - Vf(w) = \left\{1 + \Delta q - (\alpha+1)\frac{|\nabla q|^2}{q+(T-t)}\right\}\frac{\alpha K}{[q+(T-t)]^{\alpha+1}} - Vf(w).$$

Noting that

$$f(w)[q+(T-t)]^{\alpha+1} \leq \ 2CK^p$$

for K sufficiently large, it follows that

$$w_t - \Delta w - Vf(w) \geq 0 \quad \text{in } (0,T)\times B_0,$$

provided

$$1+\Delta q(x)-(\alpha+1)\frac{|\nabla q(x)|^2}{q(x)}-\frac{2C}{\alpha}K^{p-1}V(x)\geq 0\quad \text{for all } x\in B_0. \tag{4.10}$$

Fixing K and choosing r small enough, since $V(x_0)=0$, we have $(2C/\alpha)K^{p-1}V(x)<1/3$ for all $x\in B_0$. Since, by direct computation, $|\Delta q|+q^{-1}|\nabla q|^2\leq C(r)\beta$, by choosing β small enough, inequality (4.10) follows. Therefore, the comparison principle gives $u\leq w$ in $(0,T)\times B_0$ and the assertion is proved. □

Proof of Theorem 1.1. By (1.4), we have $f(s)\leq C(1+s)^p$ for all $s\geq 0$, with some $C>0$. The theorem is then a direct consequence of Lemmas 4.2 and 4.3. □

5. Blowup at zero points of the potential for weak nonlinearities

Consider the problem

$$\begin{aligned} u_t &= u_{xx}+V(x)f(u), \quad t>0,\ x\in(-1,1), && (5.1)\\ u(t,-1) &= u(t,1)\ =\ 0, \quad t>0, && (5.2)\\ u(0,x) &= u_0(x), \quad x\in(-1,1), && (5.3)\end{aligned}$$

where

$$f(u)=u[\log(1+u)]^a. \tag{5.4}$$

The following result, announced after Theorem 1.1, shows that the nonlinearity u^p cannot be replaced with a slowly growing one in Theorem 1.1. Note that our assumptions allow for instance any even, C^1 potential V such that $0\leq V\leq 1$ and $V=1$ on $[0,1/3]$. Such potential may vanish at isolated points and/or on some subintervals of $(1/3,1)$ and all its zeros will satisfy condition (1.7) whenever $V(1)>0$.

Proposition 5.1. *Assume* (5.4) *with* $1<a<2$. *Let* $V,\phi\in C^1([-1,1])$, *with* $V,\phi\geq 0$, $V(0)>0$, $\phi(0)>0$ *and* $\phi(1)=0$. *Assume that* V *and* ϕ *are even and satisfy:*

$$V',\phi'\leq 0\quad \textit{on } [0,1/3],$$

$$0\leq V(x)\leq V(1/3),\ \ 0\leq \phi(x)\leq\phi(1/3)\quad \textit{on } [1/3,1].$$

Let $u_0=\lambda\phi$ *with* $\lambda>0$ *and denote by* $u\geq 0$ *the unique, maximal classical solution of problem* (5.1)-(5.3) *and* $T=T(u_0)$ *its existence time.*

(i) Then $T<\infty$ *for all* $\lambda>0$ *sufficiently large and, whenever* $T<\infty$, *blowup is global, namely*

$$\lim_{t\to T}u(t,x)=\infty\qquad \textit{for every } x\in(-1,1). \tag{5.5}$$

(ii) Assume moreover that $\phi\in C^2([-1,1])$ *and that, for some* $r\in(0,1)$, $\phi,V>0$ *on* $[0,r]$ *and* $\phi_{xx}\geq 0$ *on* $[r,1]$. *Then, for all* $\lambda>0$ *sufficiently large, we have* $u_t\geq 0$.

We note that in the case $V = 1$, global blowup for problem (5.1)-(5.4) and $1 < a < 2$ was proved in [19] for smooth bounded domains $\Omega \subset \mathbb{R}^n$ and in [31] for $\Omega = (0, \infty)$. For radially symmetric decreasing solutions in $\Omega = \mathbb{R}^n$, global blowup as well as further qualitative properties of blow-up solutions were obtained in [10]. For the case of nonconstant, possibly vanishing potential, we will need some specific arguments. On the other hand, we obtain the following information on the blowup behavior, which shows that blowup remains of "type I" at each point $x \in (-1, 1)$, in contrast with the case of blowup at zero points of the potential for power nonlinearities (cf. cases (b) and (c) in Remark 1.2, and Lemma 4.3).

Proposition 5.2. *Let the assumptions of Proposition 5.1(ii) be in force, with $T < \infty$. Then there exist constants $C_2 \geq C_1 > 0$ such that*

$$C_1(1 - |x|) \exp\bigl[C_1(T - t)^{-1/(a-1)}\bigr] \leq u(t, x) \leq u(t, 0) \leq \exp\bigl[C_2(T - t)^{-1/(a-1)}\bigr] \tag{5.6}$$

for all $t \in (T/2, T)$ and $x \in (-1, 1)$.

We turn to the proofs of Propositions 5.1 and 5.2. Owing to our assumptions, $u(t, \cdot)$ is even for all $t \in (0, T)$, but it need not be nonincreasing on $[0, 1]$, since u_0 and V are not assumed to be so. However we shall prove that u has some partial monotonicity properties, which guarantee the persistence of the maximum at the origin, a useful fact for the proof of Propositions 5.1 and 5.2. These monotonicity properties, which rely on suitable reflection arguments, remain valid for more general problems with potential as follows. We note that the assumption $L \geq 1/3$ does not seem easy to relax.

Proposition 5.3. *Let $f : [0, \infty) \to [0, \infty)$ be of class C^1. Let $V, u_0 \in C^1([-1, 1])$, with $V, \phi \geq 0$, $V(0) > 0$, $u_0(0) > 0$ and $u_0(1) = 0$. Let $L \in [1/3, 1)$ and assume that V and ϕ are even and satisfy:*

$$V', u_0' \leq 0 \quad \text{on } [0, L], \tag{5.7}$$

$$0 \leq V(x) \leq V(L), \;\; 0 \leq u_0(x) \leq u_0(L) \quad \text{on } [L, 1]. \tag{5.8}$$

Then for any $\tau > 0$ and any classical solution u of problem (5.1)-(5.3) on $(0, \tau]$, we have

$$u_x \leq 0 \quad \text{in } (0, \tau] \times [0, L], \tag{5.9}$$

and

$$m(t) := \max_{x \in [-1,1]} u(t, x) = u(t, 0), \quad 0 < t < \tau. \tag{5.10}$$

Proof. Set $J = (-L, 1 - 2L)$. We first claim that

$$u(t, x) \geq u(t, x + 2L) \quad \text{for all } t \in [0, \tau) \text{ and } x \in J. \tag{5.11}$$

Consider the function $w(t, x) = u(t, x) - u(t, x + 2L)$. We have $w(t, -L) = 0$, since $u(t, \cdot)$ is even, and also $w(t, 1 - 2L) = u(t, 1 - 2L) \geq 0$. Moreover, for all $x \in J$, we have

$x \in (-L, L)$, due to $L \geq 1/3$, and $x + 2L \in (L, 1)$. Therefore, $V(x) \geq V(L) \geq V(x+2L)$ by (5.7)-(5.8), and similarly $w(0,x) = u_0(x) - u_0(x+2L) \geq 0$. In particular, w satisfies

$$w_t - w_{xx} = V(x)f(u(t,x)) - V(x+2L)f(u(t,x+2L)) \geq V(x)\big[f(u(t,x)) - f(u(t,x+2L))\big]$$

in $(0,\tau) \times J$. Claim (5.11) then follows from the maximum principle.

We next prove (5.9). By (5.11), we have $u(t, L+y) \leq u(t, y-L) = u(t, L-y)$ for all $y \in (0, 1-L)$, hence $u_x(t,L) \leq 0$ for all $t \in (0,\tau)$. Next, by (5.7), $z = u_x$ satisfies $z(0,x) \leq 0$ in $[0,L]$ and is a (strong) solution of

$$z_t - z_{xx} = V'(x)f(u) + V(x)f'(u)z \leq V(x)f'(u)z \quad \text{ in } (0,\tau) \times (0,L).$$

Since also $z(t,0) = 0$, property (5.9) follows from the maximum principle.

Now, by (5.9), for all $x \in (-L, L)$, we have $u(t,x) \leq u(t,0)$. Moreover, for all $x \in (L,1)$, we have $x - 2L \in J \subset (-L, L)$ due to $L \geq 1/3$. Consequently, by (5.11), we have $u(t,x) \leq u(t, x-2L) \leq u(t,0)$. Property (5.10) follows. □

Proof of Proposition 5.1. (i)

Step 1. *Finite time blowup and lower blowup estimate.* The fact that $T < \infty$ for all $\lambda > 0$ sufficiently large is a consequence of a standard Kaplan-type argument (see e.g. [30, Chapter 17]). More precisely, taking $\ell \in (0,1)$ and $c > 0$ such that $V, \phi \geq c$ on $[-\ell, \ell]$, one derives a differential inequality for the functional $\varphi(t) = \int_{-\ell}^{\ell} u(t,x)\cos(\pi x/2\ell)\,dx$ by using Jensen's inequality (f being convex), and one concludes $T < \infty$ by using the fact that $\int_2^\infty \frac{ds}{f(s)} < \infty$.

Next, by Proposition 5.3, we have $m(t) := \max_{x \in [-1,1]} u(t,x) = u(t,0)$. Since $u_{xx}(t,0) \leq 0$, it follows that

$$m'(t) \leq V(0)f(m(t)) \leq Cm(t)[\log m(t)]^a.$$

By integration, using $\lim_{t \to T} m(t) = \infty$, we deduce

$$u(t,0) \geq \exp\big[C(T-t)^{-1/(a-1)}\big], \quad 0 < t < T. \tag{5.12}$$

Step 2. *Comparison with a linear problem with fast boundary blowup source.*

By the maximum principle, we have $u \geq v$, where v is the solution of the linear problem:

$$\begin{aligned} v_t &= v_{xx}, \quad 0 < t < T,\ x \in (0,1), \\ v(t,0) &= u(t,0), \quad 0 < t < T, \\ v(t,1) &= 0, \quad 0 < t < T, \\ v(0,x) &= u_0(x), \quad x \in (0,1). \end{aligned}$$

The function v admits the following representation:

$$v(t,x) = \int_0^1 G(t,x;y)u_0(y)\,dy + \int_0^t \frac{\partial G}{\partial y}(t-s,x;0)u(s,0)\,ds, \quad 0 < t < T,\ x \in (0,1),$$

where G is the Dirichlet heat kernel of the interval $(0,1)$. This problem is studied in detail in [31] when $(0,1)$ is replaced by the half-line $(0,\infty)$, taking advantage of the explicit Gaussian heat kernel. Although this is not available in our case, it is known [35] that G satisfies the following sharp lower estimate:

$$G(t,x;y) \geq c_1 \min\Big(\frac{\rho(x)\rho(y)}{t},1\Big)t^{-1/2}\exp\big[-c_2|x-y|^2/t\big], \quad t\in(0,T],\ x,y\in[0,1],$$

for some constants $c_1,c_2>0$ (depending on T), where $\rho(x)=\min(x,1-x)$ is the distance to the boundary of $(-1,1)$. Consequently, for each $x\in(0,1)$, we have

$$\frac{\partial G}{\partial y}(t,x;0)=\lim_{y\to 0}\frac{G(t,x;y)}{y}\geq c_1\rho(x)t^{-3/2}\exp\big[-c_2|x|^2/t\big], \quad t\in(0,T].$$

Set $b=1/(a-1)>1$. Using (5.12), it follows that, for all $t\in(T/2,T)$ and $x\in(0,1)$,

$$\begin{aligned} v(t,x) &\geq c_1\rho(x)\int_0^t (t-s)^{-3/2}\exp\big[-c_2|x|^2(t-s)^{-1}\big]\exp\big[C(T-s)^{-b}\big]\,ds \\ &\geq c_1\rho(x)\int_{t-(T-t)}^{t-(T-t)/2}(t-s)^{-3/2}\exp\big[-c_2|x|^2(t-s)^{-1}\big]\exp\big[C(T-s)^{-b}\big]\,ds \\ &\geq \frac{c_1}{2}\rho(x)(T-t)^{-1/2}\exp\big[-2c_2|x|^2(T-t)^{-1}+2^{-b}C(T-t)^{-b}\big]. \end{aligned}$$

Therefore, since $b>1$, we obtain

$$u(t,x)\geq v(t,x)\geq C_1\rho(x)\exp\big[C_1(T-t)^{-1/(a-1)}\big], \quad t\in(T/2,T),\ x\in(0,1). \tag{5.13}$$

This, along with (5.12), guarantees (5.5).

(ii) This follows from a standard maximum principle argument (see, e.g., [30, Section 52.6]) observing that $u_{0,xx}+V(x)u_0[\log(1+u_0)]^a=\lambda\big(\phi_{xx}+V(x)\phi[\log(1+\lambda\phi)]^a\big)\geq 0$ in $[-1,1]$ for $\lambda>0$ large, in view of our assumptions on V and ϕ. $\square$

Proof of Proposition 5.2. Set $J=u_t-\varepsilon f(u)$ and note that $J=0$ for $x=\pm1$. Since $u_t\geq 0$ by Proposition 5.1(ii) and $u_t\not\equiv 0$ due to $T<\infty$, it follows from the Hopf Lemma that $J(t_1,x)\geq 0$ in $(-1,1)$ for some $t_1\in[T/2,T)$, if we choose $\varepsilon>0$ sufficiently small. By the computation in (4.6) (valid for any convex $f\in C^2$) and the maximum principle, we deduce that $J\geq 0$ in $(t_1,T)\times(-1,1)$. Since $V(0)>0$, we deduce the upper estimate of $u(t,0)$ upon integration, whereas the inequality $u(t,x)\leq u(t,0)$ is a consequence of (5.10).

As for the lower estimate of $u(t,x)$ in (5.6), for $1/3\leq|x|<1$ it follows from (5.13). For $0\leq|x|<1/3$, it follows from $u(t,x)\geq u(t,1/3)$, in view of (5.9) with $L=1/3$. $\square$

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