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# ON THE FRACTIONAL LANE-EMDEN EQUATION

JUAN DÁVILA, LOUIS DUPAIGNE, AND JUNCHENG WEI

ABSTRACT. We classify solutions of finite Morse index of the fractional Lane-Emden equation

$$(-\Delta)^s u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n.$$

## 1. INTRODUCTION

Fix an integer  $n \geq 1$  and let  $p_S(n)$  denote the classical Sobolev exponent:

$$p_S(n) = \begin{cases} +\infty & \text{if } n \leq 2 \\ \frac{n+2}{n-2} & \text{if } n \geq 3 \end{cases}$$

A celebrated result of Gidas and Spruck [20] asserts that there is no positive solution to the Lane-Emden equation

$$(1.1) \quad -\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n,$$

whenever  $p \in (1, p_S(n))$ . For  $p = p_S(n)$ , the same equation is known to have (up to translation and rescaling) a unique positive solution, which is radial and explicit (see Caffarelli-Gidas-Spruck [4]). Let now  $p_c(n) > p_S(n)$  denote the Joseph-Lundgren exponent:

$$p_c(n) = \begin{cases} +\infty & \text{if } n \leq 10 \\ \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11 \end{cases}$$

This exponent can be characterized as follows: for  $p \geq p_S(n)$ , the explicit singular solution  $u_s(x) = A|x|^{-\frac{2}{p-1}}$  is unstable if and only if  $p < p_c(n)$ . It was proved by Farina [18] that (1.1) has no nontrivial finite Morse index solution whenever  $1 < p < p_c(n)$ ,  $p \neq p_S(n)$ .

Through blow-up analysis, such Liouville-type theorems imply interior regularity for solutions of a large class of semilinear elliptic equations: they are known to be equivalent to universal estimates for solutions of

$$(1.2) \quad -Lu = f(x, u, \nabla u) \quad \text{in } \Omega,$$

where  $L$  is a uniformly elliptic operator with smooth coefficients, the nonlinearity  $f$  scales like  $|u|^{p-1}u$  for large values of  $u$ , and  $\Omega$  is an open set of  $\mathbb{R}^n$ . For precise statements, see the work of Polacik, Quittner and Souplet [26] in the subcritical setting, as well as its adaptation to the supercritical case by Farina and two of the authors [11].

In the present work, we are interested in understanding whether similar results hold for equations involving a nonlocal diffusion operator, the simplest of which

is perhaps the fractional laplacian. Given  $s \in (0, 1)$ , the fractional version of the Lane-Emden equation<sup>1</sup> reads

$$(1.3) \quad (-\Delta)^s u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n.$$

Here we have assumed that  $u \in C^{2\sigma}(\mathbb{R}^n)$ ,  $\sigma > s$  and

$$(1.4) \quad \int_{\mathbb{R}^n} \frac{|u(y)|}{(1+|y|)^{n+2s}} dy < +\infty,$$

so that the fractional laplacian of  $u$

$$(-\Delta)^s u(x) := \mathcal{A}_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy$$

is well-defined (in the principal-value sense) at every point  $x \in \mathbb{R}^n$ . The normalizing constant  $\mathcal{A}_{n,s} = \frac{2^{2s-1} \Gamma(\frac{n+2s}{2})}{\pi^{n/2} |\Gamma(-s)|}$  is of the order of  $s(1-s)$  as  $s$  converges to either 0 or 1.

The aforementioned classification results of Gidas-Spruck and Caffarelli-Gidas-Spruck have been generalized to the fractional setting (see Y. Li [24] and Chen-Li-Ou [8]). The corresponding fractional Sobolev exponent is given by

$$p_S(n) = \begin{cases} +\infty & \text{if } n \leq 2s \\ \frac{n+2s}{n-2s} & \text{if } n > 2s \end{cases}$$

Our main result is the following Liouville-type theorem for the fractional Lane-Emden equation.

**Theorem 1.1.** *Assume that  $n \geq 1$  and  $0 < s < \sigma < 1$ . Let  $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1+|y|)^{n+2s} dy)$  be a solution to (1.3) which is stable outside a compact set i.e. there exists  $R_0 \geq 0$  such that for every  $\varphi \in C_c^1(\mathbb{R}^n \setminus \overline{B_{R_0}})$ ,*

$$(1.5) \quad p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 dx \leq \|\varphi\|_{\dot{H}^s(\mathbb{R}^n)}^2.$$

- If  $1 < p < p_S(n)$  or if  $p_S(n) < p$  and

$$(1.6) \quad p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1}) \Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1}) \Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2},$$

then  $u \equiv 0$ ;

- If  $p = p_S(n)$ , then  $u$  has finite energy i.e.

$$\|u\|_{\dot{H}^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |u|^{p+1} < +\infty.$$

If in addition  $u$  is stable, then in fact  $u \equiv 0$ .

*Remark 1.* For  $p > p_S(n)$ , the function

$$(1.7) \quad u_s(x) = A|x|^{-\frac{2s}{p-1}}$$

<sup>1</sup>Unlike local diffusion operators, local elliptic regularity for equations of the form (1.2) where this time  $L$  is the generator of a general Markov diffusion, cannot be captured from the sole understanding of the fractional Lane-Emden equation. For example, further investigations will be needed for operators of Lévy symbol  $\psi(\xi) = \int_{S^{n-1}} |\omega \cdot \xi|^{2s} \mu(d\omega)$ , where  $\mu$  is any finite symmetric measure on the sphere  $S^{n-1}$ .

where

$$A^{p-1} = \lambda \left( \frac{n-2s}{2} - \frac{2s}{p-1} \right)$$

and where

$$(1.8) \quad \lambda(\alpha) = 2^{2s} \frac{\Gamma(\frac{n+2s+2\alpha}{4})\Gamma(\frac{n+2s-2\alpha}{4})}{\Gamma(\frac{n-2s-2\alpha}{4})\Gamma(\frac{n-2s+2\alpha}{4})}$$

is a singular solution to (1.3) (see the work by Montenegro and two of the authors [12] for the case  $s = 1/2$ , and the work by Fall [16, Lemma 3.1] for the general case). In virtue of the following Hardy inequality (due to Herbst [22])

$$\Lambda_{n,s} \int_{\mathbb{R}^n} \frac{\phi^2}{|x|^{2s}} dx \leq \|\phi\|_{H^s(\mathbb{R}^n)}^2$$

with optimal constant given by

$$\Lambda_{n,s} = 2^{2s} \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2},$$

$u_s$  is unstable if only if (1.6) holds. Let us also mention that regular radial solutions in the case  $s = 1/2$  were constructed by Chipot, Chlebik and Shafrir [9]. Recently, Harada [21] proved that for  $s = 1/2$ , condition (1.6) is the dividing line for the asymptotic behavior of radial solutions to (1.3), extending thereby the classical results of Joseph and Lundgren [23] to the fractional Lane-Emden equation in the case  $s = 1/2$ . A similar technique as in [9] allows us to show that the condition (1.6) is optimal. Indeed we have:

**Theorem 1.2.** *Assume  $p > p_S(n)$  and that (1.6) fails. Then there are radial smooth solutions  $u > 0$  with  $u(r) \rightarrow 0$  as  $r \rightarrow \infty$  of (1.3) that are stable.*

It is by now standard knowledge that the fractional laplacian can be seen as a Dirichlet-to-Neumann operator for a degenerate but *local* diffusion operator in the higher-dimensional half-space  $\mathbb{R}_+^{n+1}$ :

**Theorem 1.3** ([5, 25, 28]). *Take  $s \in (0, 1)$ ,  $\sigma > s$  and  $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1 + |y|)^{n+2s} dy)$ . For  $X = (x, t) \in \mathbb{R}_+^{n+1}$ , let*

$$\bar{u}(X) = \int_{\mathbb{R}^n} P(X, y)u(y) dy,$$

where

$$P(X, y) = p_{n,s} t^{2s} |X - y|^{-(n+2s)}$$

and  $p_{n,s}$  is chosen so that  $\int_{\mathbb{R}^n} P(X, y) dy = 1$ . Then,  $\bar{u} \in C^2(\mathbb{R}_+^{n+1}) \cap C(\overline{\mathbb{R}_+^{n+1}})$ ,  $t^{1-2s} \partial_t \bar{u} \in C(\overline{\mathbb{R}_+^{n+1}})$  and

$$\begin{cases} \nabla \cdot (t^{1-2s} \nabla \bar{u}) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \bar{u} = u & \text{on } \partial \mathbb{R}_+^{n+1}, \\ -\lim_{t \rightarrow 0} t^{1-2s} \partial_t \bar{u} = \kappa_s (-\Delta)^s u & \text{on } \partial \mathbb{R}_+^{n+1}, \end{cases}$$

where

$$(1.9) \quad \kappa_s = \frac{\Gamma(1-s)}{2^{2s-1} \Gamma(s)}.$$

Applying Theorem 1.3 to a solution of the fractional Lane-Emden equation, we end up with the equation

$$(1.10) \quad \begin{cases} -\nabla \cdot (t^{1-2s} \nabla \bar{u}) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ -\lim_{t \rightarrow 0} t^{1-2s} \partial_t \bar{u} = \kappa_s |\bar{u}|^{p-1} \bar{u} & \text{on } \partial \mathbb{R}_+^{n+1} \end{cases}$$

An essential ingredient in the proof of Theorem 1.1 is the following monotonicity formula

**Theorem 1.4.** *Take a solution to (1.10)  $\bar{u} \in C^2(\mathbb{R}_+^{n+1}) \cap C(\overline{\mathbb{R}_+^{n+1}})$  such that  $t^{1-2s} \partial_t \bar{u} \in C(\overline{\mathbb{R}_+^{n+1}})$ . For  $x_0 \in \partial \mathbb{R}_+^{n+1}$ ,  $\lambda > 0$ , let*

$$E(\bar{u}, x_0; \lambda) = \lambda^{2s \frac{p+1}{p-1} - n} \left( \frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap B^{n+1}(x_0, \lambda)} t^{1-2s} |\nabla \bar{u}|^2 dx dt - \frac{\kappa_s}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B^{n+1}(x_0, \lambda)} |\bar{u}|^{p+1} dx \right) + \lambda^{2s \frac{p+1}{p-1} - n - 1} \frac{s}{p+1} \int_{\partial B^{n+1}(x_0, \lambda) \cap \mathbb{R}_+^{n+1}} t^{1-2s} \bar{u}^2 d\sigma.$$

Then,  $E$  is a nondecreasing function of  $\lambda$ . Furthermore,

$$\frac{dE}{d\lambda} = \lambda^{2s \frac{p+1}{p-1} - n + 1} \int_{\partial B^{n+1}(x_0, \lambda) \cap \mathbb{R}_+^{n+1}} t^{1-2s} \left( \frac{\partial \bar{u}}{\partial r} + \frac{2s}{p-1} \frac{\bar{u}}{r} \right)^2 d\sigma$$

*Remark 2.* In the above,  $B^{n+1}(x_0, \lambda)$  denotes the euclidean ball in  $\mathbb{R}^{n+1}$  centered at  $x_0$  of radius  $\lambda$ ,  $\sigma$  the  $n$ -dimensional Hausdorff measure restricted to  $\partial B^{n+1}(x_0, \lambda)$ ,  $r = |X|$  the euclidean norm of a point  $X = (x, t) \in \mathbb{R}_+^{n+1}$ , and  $\partial_r = \nabla \cdot \frac{X}{r}$  the corresponding radial derivative.

An analogous monotonicity formula has been derived by Fall and Felli [17] to obtain unique continuation results for fractional equations. Previously, Caffarelli and Silvestre derived an Almgren quotient formula for the fractional laplacian in [5] and Caffarelli, Roquejoffre and Savin [6] obtained a related monotonicity formula to study regularity of nonlocal minimal surfaces. Another monotonicity formula for fractional problems was obtained by Cabré and Sire [3] and used by Frank, Lenzmann and Silvestre [19].

The proof of Theorem 1.1 follows an approach used in our earlier work with Kelei Wang [13] (see also [29]). First we derive suitable energy estimate (Section 2) and handle the critical and subcritical cases (Section 3). In the supercritical case, we make crucial use of a monotonicity formula Theorem 1.4, proved in Section 4. Thanks to it, using a blown-down analysis (first three steps of Section 6), we prove that the blow-down limit of a given solution is homogeneous. We then exclude the existence of stable homogeneous singular solutions in the optimal range of  $p$  (Section 5). Steps 5 and 6 of Section 6) are eventually devoted to prove that the solution itself is trivial. Finally we prove Theorem 1.2 in Section 7.

## 2. ENERGY ESTIMATES

**Lemma 2.1.** *Let  $u$  be a solution to (1.3). Assume that  $u$  is stable outside some ball  $B_{R_0}^n \subset \mathbb{R}^n$ . Let  $\eta \in C_c^\infty(\mathbb{R}^n \setminus \overline{B_{R_0}^n})$  and for  $x \in \mathbb{R}^n$ , define*

$$(2.1) \quad \rho(x) = \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{n+2s}} dy$$

Then,

$$\int_{\mathbb{R}^n} |u|^{p+1} \eta^2 dx + \frac{1}{p} \|u\eta\|_{\dot{H}^s(\mathbb{R}^n)}^2 \leq \frac{\mathcal{A}_{n,s}}{p-1} \int_{\mathbb{R}^n} u^2 \rho dx.$$

*Proof.* Multiply (1.3) by  $u\eta^2$ . Then,

$$\begin{aligned} \int_{\mathbb{R}^n} |u|^{p+1} \eta^2 dx &= \int_{\mathbb{R}^n} (-\Delta)^s u u \eta^2 dx \\ &= \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(u(x)\eta(x)^2 - u(y)\eta(y)^2)}{|x-y|^{n+2s}} dx dy \\ &= \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u^2(x)\eta^2(x) - u(x)u(y)(\eta^2(x) + \eta^2(y)) + u^2(y)\eta^2(y)}{|x-y|^{n+2s}} dx dy \\ &= \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)\eta(x) - u(y)\eta(y))^2 - (\eta(x) - \eta(y))^2 u(x)u(y)}{|x-y|^{n+2s}} dx dy \\ &= \|u\eta\|_{\dot{H}^s(\mathbb{R}^n)}^2 - \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2 u(x)u(y)}{|x-y|^{n+2s}} dx dy \end{aligned}$$

Using the inequality  $2ab \leq a^2 + b^2$ , we deduce that

$$(2.2) \quad \|u\eta\|_{\dot{H}^s(\mathbb{R}^n)}^2 - \int_{\mathbb{R}^n} |u|^{p+1} \eta^2 dx \leq \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} u^2 \rho dx$$

Since  $u$  is stable, we deduce that

$$(p-1) \int_{\mathbb{R}^n} |u|^{p+1} \eta^2 dx \leq \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} u^2 \rho dx$$

Going back to (2.2), it follows that

$$\frac{1}{p} \|u\eta\|_{\dot{H}^s(\mathbb{R}^n)}^2 + \int_{\mathbb{R}^n} |u|^{p+1} \eta^2 dx \leq \frac{\mathcal{A}_{n,s}}{p-1} \int_{\mathbb{R}^n} u^2 \rho dx$$

□

**Lemma 2.2.** For  $m > n/2$  and  $x \in \mathbb{R}^n$ , let

$$(2.3) \quad \eta(x) = (1 + |x|^2)^{-m/2} \quad \text{and} \quad \rho(x) = \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{n+2s}} dy$$

Then, there exists a constant  $C = C(n, s, m) > 0$  such that

$$(2.4) \quad C^{-1} (1 + |x|^2)^{-\frac{n}{2}-s} \leq \rho(x) \leq C (1 + |x|^2)^{-\frac{n}{2}-s}.$$

*Proof.* Let us prove the upper bound first. Since  $\rho$  is a continuous function, we may always assume that  $|x| \geq 1$ . Split the integral

$$\int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{n+2s}} dy$$

in the regions  $|x-y| < |x|/2$ ,  $|x|/2 \leq |x-y| \leq 2|x|$ , and  $|x-y| > 2|x|$ . When  $|x-y| \leq |x|/2$ ,

$$|\eta(x) - \eta(y)| \leq C(1 + |x|^2)^{-\frac{m+1}{2}} |x-y|.$$

So,

$$\begin{aligned} \int_{|x-y| \leq |x|/2} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{n+2s}} dy &\leq C(1 + |x|^2)^{-m-1} \int_{|x-y| \leq |x|/2} |x-y|^{2-n-2s} dy \\ &\leq C(1 + |x|^2)^{-m-s} \leq C(1 + |x|^2)^{-\frac{n}{2}-s}. \end{aligned}$$

When  $|x|/2 \leq |x-y| \leq 2|x|$ ,

$$\begin{aligned} \int_{|x|/2 \leq |x-y| \leq 2|x|} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{n+2s}} dy &\leq C|x|^{-n-2s} \int_{|y| \leq 2|x|} (\eta(x)^2 + \eta(y)^2) dy \\ &\leq C|x|^{-n-2s} (|x|^{-2m+n} + 1) \leq C(1 + |x|^2)^{-\frac{n}{2}-s}, \end{aligned}$$

where we used the assumption  $m > \frac{n}{2}$ . When  $|x-y| > 2|x|$ , then  $|y| \geq |x|$  and  $\eta(y) \leq C(1 + |x|^2)^{-m/2}$ . Then,

$$\begin{aligned} \int_{|x-y| > 2|x|} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{n+2s}} dy &\leq C(1 + |x|^2)^{-m} \int_{|x-y| > 2|x|} \frac{1}{|x-y|^{n+2s}} dy \\ &\leq C(1 + |x|^2)^{-m-s} \leq C(1 + |x|^2)^{-\frac{n}{2}-s}. \end{aligned}$$

Let us turn to the lower bound. Again, we may always assume that  $|x| \geq 1$ . Then,

$$\rho(x) \geq \int_{|y| \leq 1/2} \frac{(\eta(y) - \eta(x))^2}{|x-y|^{n+2s}} dy \geq \left(\frac{|x|}{2}\right)^{-(n+2s)} \int_{|y| \leq 1/2} (\eta(y) - 2^{-m/2})^2 dy$$

and the estimate follows.  $\square$

**Corollary 2.3.** *Let  $m > n/2$ ,  $\eta$  given by (2.3),  $R \geq R_0 \geq 1$ ,  $\psi \in C^\infty(\mathbb{R}^n)$  be such that  $0 \leq \psi \leq 1$ ,  $\psi \equiv 0$  on  $B_1^n$  and  $\psi \equiv 1$  on  $\mathbb{R}^n \setminus B_2^n$ . Let*

$$(2.5) \quad \eta_R(x) = \eta\left(\frac{x}{R}\right) \psi\left(\frac{x}{R_0}\right) \quad \text{and} \quad \rho_R(x) = \int_{\mathbb{R}^n} \frac{(\eta_R(x) - \eta_R(y))^2}{|x-y|^{n+2s}} dy$$

There exists a constant  $C = C(n, s, m, R_0) > 0$  such that for all  $|x| \geq 3R_0$

$$\rho_R(x) \leq C\eta\left(\frac{x}{R}\right)^2 |x|^{-(n+2s)} + R^{-2s} \rho\left(\frac{x}{R}\right)$$

*Proof.* Fix  $x$  such that  $|x| \geq 3R_0$ . Using the definition of  $\eta_R$  and Young's inequality, we have

$$\begin{aligned} \frac{1}{2}\rho_R(x) &\leq \eta\left(\frac{x}{R}\right)^2 \int_{\mathbb{R}^n} \frac{\left(\psi\left(\frac{x}{R_0}\right) - \psi\left(\frac{y}{R_0}\right)\right)^2}{|x-y|^{n+2s}} dy + \int_{\mathbb{R}^n} \psi\left(\frac{y}{R_0}\right)^2 \frac{(\eta\left(\frac{x}{R}\right) - \eta\left(\frac{y}{R}\right))^2}{|x-y|^{n+2s}} dy \\ &\leq \eta\left(\frac{x}{R}\right)^2 \int_{B_{2R_0}^n} \frac{1}{|x-y|^{n+2s}} dy + \int_{\mathbb{R}^n} \frac{(\eta\left(\frac{x}{R}\right) - \eta\left(\frac{y}{R}\right))^2}{|x-y|^{n+2s}} dy \\ &\leq C\eta\left(\frac{x}{R}\right)^2 |x|^{-(n+2s)} + R^{-2s} \rho\left(\frac{x}{R}\right) \end{aligned}$$

and the result follows.  $\square$

**Lemma 2.4.** *Let  $u$  be a solution of (1.3) which is stable outside a ball  $B_{R_0}^n$ . Take  $\rho_R$  as in Corollary 2.3 with  $m \in (\frac{n}{2}, \frac{n}{2} + \frac{s(p+1)}{2})$ . Then, there exists a constant  $C = C(n, s, m, p, R_0) > 0$  such that for all  $R \geq 3R_0$ ,*

$$\int_{\mathbb{R}^n} u^2 \rho_R dx \leq C \left( \int_{B_{3R_0}^n} u^2 \rho_R dx + R^{n-2s} \frac{p+1}{p-1} \right).$$

*Proof.* By Corollary 2.3, if  $R \geq |x| \geq 3R_0$ , then

$$\rho_R(x) \leq C(|x|^{-n-2s} + R^{-2s})$$

and so

$$\int_{B_R^n \setminus B_{3R_0}^n} \rho_R(x)^{\frac{p+1}{p-1}} \eta_R(x)^{-\frac{4}{p-1}} dx \leq CR^{n-2s\frac{p+1}{p-1}}.$$

Similarly, if  $|x| \geq R \geq 3R_0$ , then

$$\rho_R(x) \leq CR^{-2s} \left(1 + \frac{|x|^2}{R^2}\right)^{-\frac{n}{2}-s}$$

and so

$$\rho_R(x)^{\frac{p+1}{p-1}} \eta_R(x)^{-\frac{4}{p-1}} \leq CR^{-2s\frac{p+1}{p-1}} \left(1 + \frac{|x|^2}{R^2}\right)^{-\frac{n+2s}{2}\frac{p+1}{p-1} + \frac{2m}{p-1}}$$

Since  $m \in (\frac{n}{2}, \frac{n}{2} + s\frac{p+1}{2})$ , we have  $\frac{2m}{p-1} - \frac{n+2s}{2}\frac{p+1}{p-1} < -\frac{n}{2}$  and so

$$\int_{\mathbb{R}^n \setminus B_{3R_0}^n} \rho_R(x)^{\frac{p+1}{p-1}} \eta_R(x)^{-\frac{4}{p-1}} dx \leq CR^{n-2s\frac{p+1}{p-1}}.$$

Now,

$$\begin{aligned} \int_{\mathbb{R}^n} u^2 \rho_R dx &= \int_{B_{3R_0}^n} u^2 \rho_R dx + \int_{\mathbb{R}^n \setminus B_{3R_0}^n} u^2 \rho_R \eta_R^{-\frac{4}{p+1}} \eta_R^{\frac{4}{p+1}} dx \\ &\leq \int_{B_{3R_0}^n} u^2 \rho_R dx + \left( \int_{\mathbb{R}^n} |u|^{p+1} \eta_R^2 dx \right)^{\frac{2}{p+1}} \left( \int_{\mathbb{R}^n} \rho_R^{\frac{p+1}{p-1}} \eta_R^{-\frac{4}{p-1}} dx \right)^{\frac{p-1}{p+1}} \\ &\leq \int_{B_{3R_0}^n} u^2 \rho_R dx + CR^{(n-2s\frac{p+1}{p-1})\frac{p-1}{p+1}} \left( \int_{\mathbb{R}^n} |u|^{p+1} \eta_R^2 dx \right)^{\frac{2}{p+1}} \end{aligned}$$

By a standard approximation argument, Lemma 2.1 remains valid with  $\eta = \eta_R$  and  $\rho = \rho_R$  and so the result follows.  $\square$

**Lemma 2.5.** *Assume that  $p \neq \frac{n+2s}{n-2s}$ . Let  $u$  be a solution to (1.3) which is stable outside a ball  $B_{R_0}^n$  and  $\bar{u}$  its extension, solving (1.10). Then, there exists a constant  $C = C(n, p, s, R_0, u) > 0$  such that*

$$\int_{B_R^{n+1}} t^{1-2s} \bar{u}^2 dx dt \leq CR^{n+2(1-s)-\frac{4s}{p-1}}$$

for any  $R > 3R_0$ .

*Proof.* According to Theorem 1.3,

$$\bar{u}(x, t) = p_{n,s} \int_{\mathbb{R}^n} u(z) \frac{t^{2s}}{(|x-z|^2 + t^2)^{\frac{n+2s}{2}}} dz$$

so that

$$\bar{u}(x, t)^2 \leq p_{n,s} \int_{\mathbb{R}^n} u(z)^2 \frac{t^{2s}}{(|x-z|^2 + t^2)^{\frac{n+2s}{2}}} dz.$$

So,

$$\begin{aligned} \int_{B_R^{n+1}} t^{1-2s} \bar{u}^2 dx dt &\leq p_{n,s} \int_{|x| \leq R, z \in \mathbb{R}^n} u(z)^2 \left( \int_0^R \frac{t}{(|x-z|^2 + t^2)^{\frac{n+2s}{2}}} dt \right) dz dx \\ &\leq C \int_{|x| \leq R, z \in \mathbb{R}^n} u^2(z) \left\{ (|x-z|^2)^{-\frac{n}{2}+1-s} - (|x-z|^2 + R^2)^{-\frac{n}{2}+1-s} \right\} dz dx \end{aligned}$$



Split this last integral according to  $|x - z| < 2R$  or  $|x - z| \geq 2R$ . Then,

$$\begin{aligned} & \int_{|x| \leq R, |x-z| < 2R} u^2(z) \left\{ (|x-z|^2)^{-\frac{n}{2}+1-s} - (|x-z|^2 + R^2)^{-\frac{n}{2}+1-s} \right\} dz dx \leq \\ & \int_{|x| \leq R, |x-z| < 2R} u^2(z) (|x-z|^2)^{-\frac{n}{2}+1-s} dz dx \leq CR^{2(1-s)} \int_{B_{3R}^{n+1}} u^2(z) dz \leq \\ & CR^{2(1-s)} \left( \int |u|^{p+1} \eta_R^2 \right)^{\frac{2}{p+1}} \left( \int_{B_{3R}^{n+1}} \eta_R^{-\frac{4}{p-1}} \right)^{\frac{p-1}{p+1}} \leq \\ & CR^{2(1-s)+n\frac{p-1}{p+1}} \left( \int u^2(z) \rho_R(z) dz \right)^{\frac{2}{p+1}} \leq CR^{n+2(1-s)-\frac{4s}{p-1}}, \end{aligned}$$

where we used Hölder's inequality, then Lemma 2.1 and then Lemma 2.4. For the region  $|x - z| \geq 2R$ , the mean-value inequality implies that

$$\begin{aligned} & \int_{|x| \leq R, |x-z| \geq 2R} u^2(z) \left\{ (|x-z|^2)^{-\frac{n}{2}+1-s} - (|x-z|^2 + R^2)^{-\frac{n}{2}+1-s} \right\} dz dx \leq \\ & CR^2 \int_{|x| \leq R, |x-z| \geq 2R} u^2(z) |x-z|^{-(n+2s)} dz dx \leq CR^{n+2} \int_{|z| \geq R} u^2(z) |z|^{-(n+2s)} dz \\ & \leq CR^2 \int_{|z| \geq R} u^2 \rho dz \leq CR^{n+2(1-s)-\frac{4s}{p-1}}, \end{aligned}$$

where we used again Corollary 2.3 in the penultimate inequality and Lemma 2.4 in the last one.  $\square$

**Lemma 2.6.** *Let  $u$  be a solution to (1.3) which is stable outside a ball  $B_{R_0}^n$  and  $\bar{u}$  its extension, solving (1.10). Then, there exists a constant  $C = C(n, p, s, u) > 0$  such that*

$$\int_{B_R^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} |\nabla \bar{u}|^2 dx dt + \int_{B_R^{n+1} \cap \partial \mathbb{R}_+^{n+1}} |u|^{p+1} dx \leq CR^{n-2s\frac{p+1}{p-1}}$$

*Proof.* The  $L^{p+1}$  estimate follows from Lemmata 2.1 and 2.4. Now take a cut-off function  $\eta \in C_c^1(\mathbb{R}_+^{n+1})$  such that  $\eta = 1$  on  $\mathbb{R}_+^{n+1} \cap (B_R^{n+1} \setminus B_{2R_0}^{n+1})$  and  $\eta = 0$  on  $B_{R_0}^{n+1} \cup (\mathbb{R}_+^{n+1} \setminus B_{2R}^{n+1})$ , and multiply equation (1.10) by  $\bar{u}\eta^2$ . Then,

$$\begin{aligned} (2.6) \quad \kappa_s \int_{\partial \mathbb{R}_+^{n+1}} |\bar{u}|^{p+1} \eta^2 dx &= \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \{ \nabla \bar{u} \cdot \nabla (\bar{u}\eta^2) \} dx dt \\ &= \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \{ |\nabla(\bar{u}\eta)|^2 - \bar{u}^2 |\nabla \eta|^2 \} dx dt. \end{aligned}$$

Since  $u$  is stable outside  $B_{R_0}^{n+1}$ , so is  $\bar{u}$  and we deduce that

$$\frac{1}{p} \int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla(\bar{u}\eta)|^2 dx dt \geq \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \{ |\nabla(\bar{u}\eta)|^2 - \bar{u}^2 |\nabla \eta|^2 \} dx dt.$$

In other words,

$$(2.7) \quad p' \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \bar{u}^2 |\nabla \eta|^2 dx dt \geq \int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla(\bar{u}\eta)|^2 dx dt,$$

where  $\frac{1}{p'} + \frac{1}{p} = 1$ . We then apply Lemma 2.5.  $\square$

## 3. THE SUBCRITICAL CASE

In this section, we prove Theorem 1.1 for  $1 < p \leq p_S(n)$ .

*Proof.* Take a solution  $u$  which is stable outside some ball  $B_{R_0}^n$ . Apply Lemma 2.4 and let  $R \rightarrow +\infty$ . Since  $p \leq p_S(n)$ , we deduce that  $u \in \dot{H}^s(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$ . Multiplying the equation (1.3) by  $u$  and integrating, we deduce that

$$(3.1) \quad \int_{\mathbb{R}^n} |u|^{p+1} = \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2,$$

while multiplying by  $u^\lambda$  given for  $\lambda > 0$  and  $x \in \mathbb{R}^n$  by

$$u^\lambda(x) = u(\lambda x)$$

yields

$$\int_{\mathbb{R}^n} |u|^{p-1} u^\lambda = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} u^\lambda = \lambda^s \int_{\mathbb{R}^n} w w^\lambda,$$

where  $w = (-\Delta)^{s/2} u$ . Following Ros-Oton and Serra [27], we use the change of variable  $y = \sqrt{\lambda} x$  to deduce that

$$\lambda^s \int_{\mathbb{R}^n} w w^\lambda dx = \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w^{\sqrt{\lambda}} w^{1/\sqrt{\lambda}} dy$$

Hence,

$$\begin{aligned} -\frac{n}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} &= \int_{\mathbb{R}^n} x \cdot \nabla \frac{|u|^{p+1}}{p+1} = \int_{\mathbb{R}^n} (|u|^{p-1} u) x \cdot \nabla u = \\ &= \frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\mathbb{R}^n} |u|^{p-1} u u^\lambda = \frac{d}{d\lambda} \Big|_{\lambda=1} \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w^{\sqrt{\lambda}} w^{1/\sqrt{\lambda}} dy = \\ &= \frac{2s-n}{2} \int_{\mathbb{R}^n} w^2 + \frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\mathbb{R}^n} w^{\sqrt{\lambda}} w^{1/\sqrt{\lambda}} dy = \frac{2s-n}{2} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2 \end{aligned}$$

In the last equality, we have used the fact that  $w \in C^1(\mathbb{R}^n)$ , as follows by elliptic regularity. We have just proved the following Pohozaev identity

$$\frac{n}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} = \frac{n-2s}{2} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2$$

For  $p < p_S(n)$ , the above identity together with (3.1) force  $u \equiv 0$ . For  $p = p_S(n)$ , we are left with proving that there is no stable nontrivial solution. Since  $u \in \dot{H}^s(\mathbb{R}^n)$ , we may apply the stability inequality (1.5) with test function  $\varphi = u$ , so that

$$p \int_{\mathbb{R}^n} |u|^{p+1} \leq \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2.$$

This contradicts (3.1) unless  $u \equiv 0$ .  $\square$

In the following sections, we present several tools to study the supercritical case.

## 4. THE MONOTONICITY FORMULA

In this section, we prove Theorem 1.4.

*Proof.* Since the equation is invariant under translation, it suffices to consider the case where the center of the considered ball is the origin  $x_0 = 0$ . Let

$$(4.1) \quad E_1(\bar{u}; \lambda) = \lambda^{2s \frac{p+1}{p-1} - n} \left( \int_{\mathbb{R}_+^{n+1} \cap B_\lambda^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_\lambda^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}|^{p+1} dx \right)$$

For  $X \in \mathbb{R}_+^{n+1}$ , let also

$$(4.2) \quad U(X; \lambda) = \lambda^{\frac{2s}{p-1}} \bar{u}(\lambda X).$$

Then,  $U$  satisfies the three following properties:  $U$  solves (1.10),

$$(4.3) \quad E_1(\bar{u}; \lambda) = E_1(U; 1),$$

and, using subscripts to denote partial derivatives,

$$(4.4) \quad \lambda U_\lambda = \frac{2s}{p-1} U + r U_r.$$

Differentiating the right-hand side of (4.3), we find

$$\frac{dE_1}{d\lambda}(\bar{u}; \lambda) = \int_{\mathbb{R}_+^{n+1} \cap B_1^{n+1}} t^{1-2s} \nabla U \cdot \nabla U_\lambda dx dt - \kappa_s \int_{\partial \mathbb{R}_+^{n+1} \cap B_1^{n+1}} |U|^{p-1} U_\lambda dx.$$

Integrating by parts and then using (4.4),

$$\begin{aligned} \frac{dE_1}{d\lambda}(\bar{u}; \lambda) &= \int_{\partial B_1^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} U_r U_\lambda d\sigma \\ &= \lambda \int_{\partial B_1^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} U_\lambda^2 d\sigma - \frac{2s}{p-1} \int_{\partial B_1^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} U U_\lambda d\sigma \\ &= \lambda \int_{\partial B_1^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} U_\lambda^2 d\sigma - \frac{s}{p-1} \left( \int_{\partial B_1^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} U^2 d\sigma \right)_\lambda \end{aligned}$$

Scaling back, the theorem follows.  $\square$

## 5. HOMOGENEOUS SOLUTIONS

**Theorem 5.1.** *Let  $\bar{u}$  be a stable homogeneous solution of (1.10). Assume that  $p > \frac{n+2s}{n-2s}$  and*

$$(5.1) \quad p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1}) \Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1}) \Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}.$$

Then,  $\bar{u} \equiv 0$ .

*Proof.* Take standard polar coordinates in  $\mathbb{R}_+^{n+1}$ :  $X = (x, t) = r\theta$ , where  $r = |X|$  and  $\theta = \frac{X}{|X|}$ . Let  $\theta_1 = \frac{t}{|X|}$  denote the component of  $\theta$  in the  $t$  direction and  $S_+^n = \{X \in \mathbb{R}_+^{n+1} : r = 1, \theta_1 > 0\}$  denote the upper unit half-sphere.

**Step 1.** Let  $\bar{u}$  be a homogeneous solution of (1.10) i.e. assume that for some  $\psi \in C^2(S_+^n)$ ,

$$\bar{u}(X) = r^{-\frac{2s}{p-1}} \psi(\theta).$$

Then,

$$(5.2) \quad \int_{S_+^n} \theta_1^{1-2s} |\nabla \psi|^2 + \beta \int_{S_+^n} \theta_1^{1-2s} \psi^2 = \kappa_s \int_{\partial S_+^n} |\psi|^{p+1},$$

where  $\kappa_s$  is given by (1.9) and

$$\beta = \frac{2s}{p-1} \left( n - 2s - \frac{2s}{p-1} \right).$$

Indeed, since  $\bar{u}$  solves (1.10) and is homogeneous,  $\psi$  solves

$$(5.3) \quad \begin{cases} -\operatorname{div}(\theta_1^{1-2s} \nabla \psi) + \beta \theta_1^{1-2s} \psi = 0 & \text{on } S_+^n \\ -\lim_{\theta_1 \rightarrow 0} \theta_1^{1-2s} \partial_{\theta_1} \psi = \kappa_s |\psi|^{p-1} \psi & \text{on } \partial S_+^n, \end{cases}$$

Multiplying (5.3) by  $\psi$  and integrating, (5.2) follows.

**Step 2.** For all  $\varphi \in C^1(S_+^n)$ ,

$$(5.4) \quad \kappa_s p \int_{\partial S_+^n} |\psi|^{p-1} \varphi^2 \leq \int_{S_+^n} \theta_1^{1-2s} |\nabla \varphi|^2 + \left( \frac{n-2s}{2} \right)^2 \int_{S_+^n} \theta_1^{1-2s} \varphi^2$$

By definition,  $\bar{u}$  is stable if for all  $\phi \in C_c^1(\overline{\mathbb{R}_+^{n+1}})$ ,

$$(5.5) \quad \kappa_s p \int_{\partial \mathbb{R}_+^{n+1}} |\bar{u}|^{p-1} \phi^2 dx \leq \int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla \phi|^2 dx dt$$

Choose a standard cut-off function  $\eta_\epsilon \in C_c^1(\mathbb{R}_+^*)$  at the origin and at infinity i.e.  $\chi_{(\epsilon, 1/\epsilon)}(r) \leq \eta_\epsilon(r) \leq \chi_{(\epsilon/2, 2/\epsilon)}(r)$ . Let also  $\varphi \in C^1(S_+^n)$ , apply (5.5) with

$$\phi(X) = r^{-\frac{n-2s}{2}} \eta_\epsilon(r) \varphi(\theta) \quad \text{for } X \in \mathbb{R}_+^{n+1},$$

and let  $\epsilon \rightarrow 0$ . Inequality (5.4) follows.

**Step 3.** For  $\alpha \in (0, \frac{n-2s}{2})$ ,  $x \in \mathbb{R}^n \setminus \{0\}$ , let

$$v_\alpha(x) = |x|^{-\frac{n-2s}{2} + \alpha}$$

and  $\bar{v}_\alpha$  its extension, as defined in Theorem 1.3. Then,  $\bar{v}_\alpha$  is homogeneous i.e. there exists  $\phi_\alpha \in C^2(S_+^n)$  such that for  $X \in \mathbb{R}_+^{n+1} \setminus \{0\}$ ,

$$\bar{v}_\alpha(X) = r^{-\frac{n-2s}{2} + \alpha} \phi_\alpha(\theta).$$

In addition, for all  $\varphi \in C^1(S_+^n)$ ,

$$(5.6) \quad \begin{aligned} \int_{S_+^n} \theta_1^{1-2s} |\nabla \varphi|^2 + \left( \left( \frac{n-2s}{2} \right)^2 - \alpha^2 \right) \int_{S_+^n} \theta_1^{1-2s} \varphi^2 \\ = \kappa_s \lambda(\alpha) \int_{\partial S_+^n} \varphi^2 + \int_{S_+^n} \theta_1^{1-2s} \phi_\alpha^2 \left| \nabla \left( \frac{\varphi}{\phi_\alpha} \right) \right|^2 \end{aligned}$$

Indeed, according to Fall [16, Lemma 3.1],  $\bar{v}_\alpha$  is homogeneous. Using the calculus identity stated by Fall-Felli in [17, Lemma 2.1], we get

$$(5.7) \quad \begin{cases} -\operatorname{div}(\theta_1^{1-2s} \nabla \phi_\alpha) + \left( \left( \frac{n-2s}{2} \right)^2 - \alpha^2 \right) \theta_1^{1-2s} \phi_\alpha = 0 & \text{on } S_+^n \\ \phi_\alpha = 1 & \text{on } \partial S_+^n. \end{cases}$$

Multiply equation (5.7) by  $\varphi^2/\phi_\alpha$ , integrate by parts, apply the calculus identity

$$\nabla\phi_\alpha \cdot \nabla \frac{\varphi^2}{\phi_\alpha} = |\nabla\varphi|^2 - \left| \nabla \frac{\varphi}{\phi_\alpha} \right|^2 \phi_\alpha^2$$

and recall from Fall [16, Lemma 3.1] that

$$-\lim_{t \rightarrow 0} t^{1-2s} \partial_t \bar{v}_\alpha = \kappa_s \lambda(\alpha) |x|^{-\frac{n-2s}{2} + \alpha - 2s},$$

where  $\lambda(\alpha)$  is given by (1.8).

**Step 4.** For  $\alpha \in (0, \frac{n-2s}{2})$

$$(5.8) \quad \phi_0 \leq \phi_\alpha \quad \text{on } S_+^n.$$

Indeed, on  $S_+^n$ ,

$$\operatorname{div}(\theta_1^{1-2s} \nabla \phi_0) = \left( \frac{n-2s}{2} \right)^2 \theta_1^{1-2s} \phi_0 \geq \left( \left( \frac{n-2s}{2} \right)^2 - \alpha^2 \right) \theta_1^{1-2s} \phi_0$$

so  $\phi_0$  is a sub-solution of (5.7). By the maximum principle, the conclusion follows.

**Step 5.** End of proof. Fix  $\alpha \in (0, \frac{n-2s}{2})$  given by

$$\alpha = \frac{n-2s}{2} - \frac{2s}{p-1}$$

so that

$$\left( \frac{n-2s}{2} \right)^2 - \alpha^2 = \frac{2s}{p-1} \left( n-2s - \frac{2s}{p-1} \right) = \beta,$$

where  $\beta$  is the constant appearing in (5.3).

Use the stability inequality (5.4) with  $\varphi = \frac{\psi\phi_0}{\phi_\alpha}$ :

$$(5.9) \quad \kappa_s p \int_{\partial S_+^n} |\psi|^{p+1} \leq \int_{S_+^n} \theta_1^{1-2s} \left| \nabla \left( \frac{\psi\phi_0}{\phi_\alpha} \right) \right|^2 + \left( \frac{n-2s}{2} \right)^2 \int_{S_+^n} \theta_1^{1-2s} \left( \frac{\psi\phi_0}{\phi_\alpha} \right)^2.$$

Note that a particular case of the identity (5.6) is

$$(5.10) \quad \int_{S_+^n} \theta_1^{1-2s} |\nabla\varphi|^2 + \left( \frac{n-2s}{2} \right)^2 \int_{S_+^n} \theta_1^{1-2s} \varphi^2 = \kappa_s \Lambda_{n,s} \int_{\partial S_+^n} \varphi^2 + \int_{S_+^n} \theta_1^{1-2s} \phi_0^2 \left| \nabla \left( \frac{\varphi}{\phi_0} \right) \right|^2$$

Using (5.10) (with  $\varphi = \frac{\psi\phi_0}{\phi_\alpha}$ ), (5.9) becomes

$$\kappa_s p \int_{\partial S_+^n} |\psi|^{p+1} \leq \kappa_s \Lambda_{n,s} \int_{\partial S_+^n} \psi^2 + \int_{S_+^n} \theta_1^{1-2s} \phi_0^2 \left| \nabla \left( \frac{\psi}{\phi_\alpha} \right) \right|^2.$$

By (5.8), we deduce that

$$\kappa_s p \int_{\partial S_+^n} |\psi|^{p+1} \leq \kappa_s \Lambda_{n,s} \int_{\partial S_+^n} \psi^2 + \int_{S_+^n} \theta_1^{1-2s} \phi_\alpha^2 \left| \nabla \left( \frac{\psi}{\phi_\alpha} \right) \right|^2.$$

Using again the identity (5.6), we deduce that

$$\kappa_s p \int_{\partial S_+^n} |\psi|^{p+1} \leq \kappa_s (\Lambda_{n,s} - \lambda(\alpha)) \int_{\partial S_+^n} \psi^2 + \int_{S_+^n} \theta_1^{1-2s} |\nabla\psi|^2 + \beta \int_{S_+^n} \theta_1^{1-2s} \psi^2$$

Comparing with (5.2), it follows that

$$(5.11) \quad (p-1) \int_{\partial S_+^n} |\psi|^{p+1} \leq (\Lambda_{n,s} - \lambda(\alpha)) \int_{\partial S_+^n} \psi^2.$$

But from (5.2) and (5.6)

$$\int_{\partial S_+^n} |\psi|^{p+1} \geq \lambda(\alpha) \int_{\partial S_+^n} \psi^2$$

Combined with (5.11), we find that

$$\lambda(\alpha)p \leq \Lambda_{n,s}$$

unless  $\psi \equiv 0$ . □

## 6. BLOW-DOWN ANALYSIS

*Proof of Theorem 1.1.* Assume that  $p > p_S(n)$ . Take a solution  $u$  of (1.3) which is stable outside the ball of radius  $R_0$  and let  $\bar{u}$  be its extension solving (1.10).

**Step 1.**  $\lim_{\lambda \rightarrow +\infty} E(\bar{u}, 0; \lambda) < +\infty$ .

Since  $E$  is nondecreasing, it suffices to show that  $E(\bar{u}, 0; \lambda)$  is bounded. Write  $E = E_1 + E_2$ , where  $E_1$  is given by (4.1) and

$$E_2(\bar{u}; \lambda) = \lambda^{2s \frac{p+1}{p-1} - n - 1} \frac{s}{p+1} \int_{\partial B^{n+1}(0, \lambda) \cap \mathbb{R}_+^{n+1}} t^{1-2s} \bar{u}^2 d\sigma$$

By Lemma 2.6,  $E_1$  is bounded. Since  $E$  is nondecreasing,

$$E(\bar{u}; \lambda) \leq \frac{1}{\lambda} \int_{\lambda}^{2\lambda} E(u; t) dt \leq C + \lambda^{2s \frac{p+1}{p-1} - n - 1} \int_{B_{2\lambda}^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} \bar{u}^2.$$

Applying Lemma 2.5, we deduce that  $E$  is bounded.

**Step 2.** There exists a sequence  $\lambda_i \rightarrow +\infty$  such that  $(\bar{u}^{\lambda_i})$  converges weakly in  $H_{loc}^1(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$  to a function  $\bar{u}^\infty$ .

This follows from the fact that  $(\bar{u}^{\lambda_i})$  is bounded in  $H_{loc}^1(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$  by Lemma 2.6.

**Step 3.**  $\bar{u}^\infty$  is homogeneous

To see this, apply the scale invariance of  $E$ , its finiteness and the monotonicity formula: given  $R_2 > R_1 > 0$ ,

$$\begin{aligned} 0 &= \lim_{i \rightarrow +\infty} E(\bar{u}; \lambda_i R_2) - E(\bar{u}; \lambda_i R_1) \\ &= \lim_{i \rightarrow +\infty} E(\bar{u}^{\lambda_i}; R_2) - E(\bar{u}^{\lambda_i}; R_1) \\ &\geq \liminf_{i \rightarrow +\infty} \int_{(B_{R_2}^{n+1} \setminus B_{R_1}^{n+1}) \cap \mathbb{R}_+^{n+1}} t^{1-2s} r^{2-n+\frac{4s}{p-1}} \left( \frac{2s}{p-1} \frac{\bar{u}^{\lambda_i}}{r} + \frac{\partial \bar{u}^{\lambda_i}}{\partial r} \right)^2 dx dt \\ &\geq \int_{(B_{R_2}^{n+1} \setminus B_{R_1}^{n+1}) \cap \mathbb{R}_+^{n+1}} t^{1-2s} r^{2-n+\frac{4s}{p-1}} \left( \frac{2s}{p-1} \frac{\bar{u}^\infty}{r} + \frac{\partial \bar{u}^\infty}{\partial r} \right)^2 dx dt \end{aligned}$$

Note that in the last inequality we only used the weak convergence of  $(\bar{u}^{\lambda_i})$  to  $\bar{u}^\infty$  in  $H_{loc}^1(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$ . So,

$$\frac{2s}{p-1} \frac{\bar{u}^\infty}{r} + \frac{\partial \bar{u}^\infty}{\partial r} = 0 \quad a.e. \text{ in } \mathbb{R}_+^{n+1}.$$

And so,  $u^\infty$  is homogeneous.

**Step 4.**  $\bar{u}^\infty \equiv 0$

Simply apply Theorem 5.1.

**Step 5.**  $(\bar{u}^{\lambda_i})$  converges strongly to zero in  $H^1(B_R^{n+1} \setminus B_\epsilon^{n+1}; t^{1-2s} dx dt)$  and  $(u^{\lambda_i})$  converges strongly to zero in  $L^{p+1}(B_R^{n+1} \setminus B_\epsilon^{n+1})$  for all  $R > \epsilon > 0$ . Indeed, by Steps 2 and 3,  $(\bar{u}^{\lambda_i})$  is bounded in  $H_{loc}^1(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$  and converges weakly to 0. It follows that  $(\bar{u}^{\lambda_i})$  converges strongly to 0 in  $L_{loc}^2(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$ . Indeed, by the standard Rellich-Kondrachov theorem and a diagonal argument, passing to a subsequence we obtain

$$\int_{\mathbb{R}_+^{n+1} \cap (B_R^{n+1} \setminus A)} t^{1-2s} |\bar{u}^{\lambda_i}|^2 dx dt \rightarrow 0,$$

as  $i \rightarrow \infty$ , for any  $B_R^{n+1} = B_R^{n+1}(0) \subset \mathbb{R}^{n+1}$  and  $A$  of the form  $A = \{(x, t) \in \mathbb{R}_+^{n+1} : 0 < t < r/2\}$ , where  $R, r > 0$ . By [15, Theorem 1.2],

$$\int_{\mathbb{R}_+^{n+1} \cap B_r^{n+1}(x)} t^{1-2s} |\bar{u}^{\lambda_i}|^2 dx dt \leq Cr^2 \int_{\mathbb{R}_+^{n+1} \cap B_r^{n+1}(x)} t^{1-2s} |\nabla \bar{u}^{\lambda_i}|^2 dx dt$$

for any  $x \in \partial \mathbb{R}_+^{n+1}$ ,  $|x| \leq R$ , with a uniform constant  $C$ . Covering  $B_R^{n+1} \cap A$  with half balls  $B_r^{n+1}(x) \cap \mathbb{R}_+^{n+1}$ ,  $x \in \partial \mathbb{R}_+^{n+1}$  with finite overlap, we see that

$$\int_{B_R^{n+1} \cap A} t^{1-2s} |\bar{u}^{\lambda_i}|^2 dx dt \leq Cr^2 \int_{B_R^{n+1} \cap A} t^{1-2s} |\nabla \bar{u}^{\lambda_i}|^2 dx dt \leq Cr^2,$$

and from this we conclude that  $(\bar{u}^{\lambda_i})$  converges strongly to 0 in  $L_{loc}^2(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$ .

Now, using (2.7),  $(\bar{u}^{\lambda_i})$  converges strongly to 0 in  $H_{loc}^1(\mathbb{R}_+^{n+1} \setminus \{0\}; t^{1-2s} dx dt)$  and by (2.6), the convergence also holds in  $L_{loc}^{p+1}(\mathbb{R}^n \setminus \{0\})$ .

**Step 6.**  $\bar{u} \equiv 0$ .

Indeed,

$$\begin{aligned} E_1(\bar{u}; \lambda) &= E_1(\bar{u}^\lambda; 1) = \int_{\mathbb{R}_+^{n+1} \cap B_1^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}^\lambda|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}^\lambda|^{p+1} dx \\ &= \int_{\mathbb{R}_+^{n+1} \cap B_\epsilon^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}^\lambda|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_\epsilon^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}^\lambda|^{p+1} dx + \\ &\quad \int_{\mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}^\lambda|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}^\lambda|^{p+1} dx \\ &= \epsilon^{n-2s} \frac{p+1}{p-1} E_1(\bar{u}; \lambda \epsilon) + \int_{\mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}^\lambda|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}^\lambda|^{p+1} dx \\ &\leq C \epsilon^{n-2s} \frac{p+1}{p-1} + \int_{\mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}^\lambda|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}^\lambda|^{p+1} dx \end{aligned}$$

Letting  $\lambda \rightarrow +\infty$  and then  $\epsilon \rightarrow 0$ , we deduce that  $\lim_{\lambda \rightarrow +\infty} E_1(\bar{u}; \lambda) = 0$ . Using the monotonicity of  $E$ ,

$$E(\bar{u}; \lambda) \leq \frac{1}{\lambda} \int_\lambda^{2\lambda} E(t) dt \leq \sup_{[\lambda, 2\lambda]} E_1 + C \lambda^{-n-1+2s} \frac{p+1}{p-1} \int_{B_{2\lambda}^{n+1} \setminus B_\lambda^{n+1}} \bar{u}^2$$

and so  $\lim_{\lambda \rightarrow +\infty} E(\bar{u}; \lambda) = 0$ . Since  $u$  is smooth, we also have  $E(\bar{u}; 0) = 0$ . Since  $E$  is monotone,  $E \equiv 0$  and so  $\bar{u}$  must be homogeneous, a contradiction unless  $\bar{u} \equiv 0$ .  $\square$

## 7. CONSTRUCTION OF RADIAL ENTIRE STABLE SOLUTIONS

Let  $\bar{u}_s$  denote the extension of the singular solution  $u_s$  (1.7) to  $\mathbb{R}_+^{n+1}$  defined by

$$\bar{u}_s(X) = \int_{\mathbb{R}^n} P(X, y) u(y) dy.$$

Let  $B_1^{n+1}$  denote the unit ball in  $\mathbb{R}^{n+1}$  and for  $\lambda \geq 0$ , consider

$$(7.1) \quad \begin{cases} \operatorname{div}(t^{1-2s} \nabla u) = 0 & \text{in } B_1^{n+1} \cap \mathbb{R}_+^{n+1} \\ u = \lambda \bar{u}_s & \text{on } \partial B_1^{n+1} \cap \mathbb{R}_+^{n+1} \\ -\lim_{t \rightarrow 0} (t^{1-2s} u_t) = \kappa_s u^p & \text{on } B_1^{n+1} \cap \{t = 0\}. \end{cases}$$

Take  $\lambda \in (0, 1)$ . Since  $u_s$  is a positive supersolution of (7.1), there exists a minimal solution  $u = u_\lambda$ . By minimality, the family  $(u_\lambda)$  is nondecreasing and  $u_\lambda$  is axially symmetric, that is,  $u_\lambda(x, t) = u_\lambda(r, t)$  with  $r = |x| \in [0, 1]$ . In addition, for a fixed value  $\lambda \in (0, 1)$ ,  $u_\lambda$  is bounded, as can be proved by the truncation method of [1], see also [10] and radially decreasing by the moving plane method (see [7] for a similar setting). From now on let us assume that  $p_S(n) < p$  and

$$p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1}) \Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1}) \Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} \leq \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2},$$

which means that the singular solution  $u_s$  is stable. Then,  $u_\lambda \uparrow u_s$  as  $\lambda \uparrow 1$ , using the classical convexity argument in [2] (see also Section 3.2.2 in [14]). Let  $\lambda_j \uparrow 1$  and

$$m_j = \|u_{\lambda_j}\|_{L^\infty} = u_{\lambda_j}(0), \quad R_j = m_j^{\frac{p-1}{2s}},$$

so that  $m_j, R_j \rightarrow \infty$  as  $j \rightarrow \infty$ . Set

$$v_j(x) = m_j^{-1} u_{\lambda_j}(x/R_j).$$

Then  $0 \leq v_j \leq 1$  is a bounded solution of

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla v_j) = 0 & \text{in } B_{R_j}^{n+1} \cap \mathbb{R}_+^{n+1} \\ v_j = \lambda_j \bar{u}_s & \text{on } \partial B_{R_j}^{n+1} \cap \mathbb{R}_+^{n+1} \\ -\lim_{t \rightarrow 0} (t^{1-2s} (v_j)_t) = \kappa_s v_j^p & \text{on } B_{R_j}^{n+1} \cap \{t = 0\}. \end{cases}$$

Moreover  $v_j \leq \bar{u}_s$  in  $B_{R_j}^{n+1} \cap \mathbb{R}_+^{n+1}$  and  $v_j(0) = 1$ . Using elliptic estimates we find (for a subsequence) that  $(v_j)$  converges uniformly on compact sets of  $\overline{\mathbb{R}_+^{n+1}}$  to a function  $v$  that is axially symmetric and solves

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla v) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ -\lim_{t \rightarrow 0} (t^{1-2s} v_t) = \kappa_s v^p & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Moreover  $0 \leq v \leq 1$ ,  $v(0) = 1$  and  $v \leq \bar{u}_s$ . This  $v$  restricted to  $\mathbb{R}^n \times \{0\}$  is a radial, bounded, smooth solution of (1.3) and from  $v \leq \bar{u}_s$  we deduce that  $v$  is stable.

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