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Abstract. We classify solutions of finite Morse index of the fractional Lane-Emden equation

$$(-\Delta)^s u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n.$$

1. Introduction

Fix an integer $n \geq 1$ and let $p_S(n)$ denote the classical Sobolev exponent:

$$p_S(n) = \begin{cases} +\infty & \text{if } n \leq 2 \\ \dfrac{n+2}{n-2} & \text{if } n \geq 3 \end{cases}$$

A celebrated result of Gidas and Spruck [20] asserts that there is no positive solution to the Lane-Emden equation

$$-\Delta u = |u|^{p-1}u \qquad \text{in } \mathbb{R}^n, \tag{1.1}$$

whenever $p \in (1, p_S(n))$. For $p = p_S(n)$, the same equation is known to have (up to translation and rescaling) a unique positive solution, which is radial and explicit (see Caffarelli-Gidas-Spruck [4]). Let now $p_c(n) > p_S(n)$ denote the Joseph-Lundgren exponent:

$$p_c(n) = \begin{cases} +\infty & \text{if } n \leq 10 \\ \dfrac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11 \end{cases}$$

This exponent can be characterized as follows: for $p \geq p_S(n)$, the explicit singular solution $u_s(x) = A|x|^{-\frac{2}{p-1}}$ is unstable if and only if $p < p_c(n)$. It was proved by Farina [18] that (1.1) has no nontrivial finite Morse index solution whenever $1 < p < p_c(n)$, $p \neq p_S(n)$.

Through blow-up analysis, such Liouville-type theorems imply interior regularity for solutions of a large class of semilinear elliptic equations: they are known to be equivalent to universal estimates for solutions of

$$-Lu = f(x, u, \nabla u) \quad \text{in } \Omega, \tag{1.2}$$

where L is a uniformly elliptic operator with smooth coefficients, the nonlinearity f scales like $|u|^{p-1}u$ for large values of u, and Ω is an open set of $\mathbb{R}^n$. For precise statements, see the work of Polacik, Quittner and Souplet [26] in the subcritical setting, as well as its adaptation to the supercritical case by Farina and two of the authors [11].

In the present work, we are interested in understanding whether similar results hold for equations involving a nonlocal diffusion operator, the simplest of which is perhaps the fractional laplacian. Given $s \in (0,1)$, the fractional version of the Lane-Emden equation[1] reads

$$(-\Delta)^s u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n. \tag{1.3}$$

Here we have assumed that $u \in C^{2\sigma}(\mathbb{R}^n)$, $\sigma > s$ and

$$\int_{\mathbb{R}^n} \frac{|u(y)|}{(1+|y|)^{n+2s}}\,dy < +\infty, \tag{1.4}$$

so that the fractional laplacian of u

$$(-\Delta)^s u(x) := \mathcal{A}_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy$$

is well-defined (in the principal-value sense) at every point $x \in \mathbb{R}^n$. The normalizing constant $\mathcal{A}_{n,s} = \frac{2^{2s-1}}{\pi^{n/2}} \frac{\Gamma\left(\frac{n+2s}{2}\right)}{|\Gamma(-s)|}$ is of the order of $s(1-s)$ as s converges to either 0 or 1.

The aforementioned classification results of Gidas-Spruck and Caffarelli-Gidas-Spruck have been generalized to the fractional setting (see Y. Li [24] and Chen-Li-Ou [8]). The corresponding fractional Sobolev exponent is given by

$$p_S(n) = \begin{cases} +\infty & \text{if } n \le 2s \\ \dfrac{n+2s}{n-2s} & \text{if } n > 2s \end{cases}$$

Our main result is the following Liouville-type theorem for the fractional Lane-Emden equation.

Theorem 1.1. *Assume that $n \ge 1$ and $0 < s < \sigma < 1$. Let $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1+|y|)^{n+2s}dy)$ be a solution to (1.3) which is stable outside a compact set i.e. there exists $R_0 \ge 0$ such that for every $\varphi \in C^1_c(\mathbb{R}^n \setminus \overline{B_{R_0}})$,*

$$p \int_{\mathbb{R}^n} |u|^{p-1}\varphi^2\,dx \le \|\varphi\|^2_{\dot{H}^s(\mathbb{R}^n)}. \tag{1.5}$$

- *If $1 < p < p_S(n)$ or if $p_S(n) < p$ and*

$$p\frac{\Gamma(\frac{n}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}, \tag{1.6}$$

  then $u \equiv 0$;
- *If $p = p_S(n)$, then u has finite energy i.e.*

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |u|^{p+1} < +\infty.$$

  If in addition u is stable, then in fact $u \equiv 0$.

Remark 1. For $p > p_S(n)$, the function

$$u_s(x) = A|x|^{-\frac{2s}{p-1}} \tag{1.7}$$

[1] Unlike local diffusion operators, local elliptic regularity for equations of the form (1.2) where this time L is the generator of a general Markov diffusion, cannot be captured from the sole understanding of the fractional Lane-Emden equation. For example, further investigations will be needed for operators of Lévy symbol $\psi(\xi) = \int_{S^{n-1}} |\omega \cdot \xi|^{2s}\mu(d\omega)$, where μ is any finite symmetric measure on the sphere S^{n-1}.

where

$$A^{p-1} = \lambda\left(\frac{n-2s}{2} - \frac{2s}{p-1}\right)$$

and where

$$\lambda(\alpha) = 2^{2s}\frac{\Gamma(\frac{n+2s+2\alpha}{4})\Gamma(\frac{n+2s-2\alpha}{4})}{\Gamma(\frac{n-2s-2\alpha}{4})\Gamma(\frac{n-2s+2\alpha}{4})} \tag{1.8}$$

is a singular solution to (1.3) (see the work by Montenegro and two of the authors [12] for the case $s = 1/2$, and the work by Fall [16, Lemma 3.1] for the general case). In virtue of the following Hardy inequality (due to Herbst [22])

$$\Lambda_{n,s}\int_{\mathbb{R}^n}\frac{\phi^2}{|x|^{2s}}\,dx \le \|\phi\|^2_{\dot{H}^s(\mathbb{R}^n)}$$

with optimal constant given by

$$\Lambda_{n,s} = 2^{2s}\frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2},$$

u_s is unstable if only if (1.6) holds. Let us also mention that regular radial solutions in the case $s = 1/2$ were constructed by Chipot, Chlebik ad Shafrir [9]. Recently, Harada [21] proved that for $s = 1/2$, condition (1.6) is the dividing line for the asymptotic behavior of radial solutions to (1.3), extending thereby the classical results of Joseph and Lundgren [23] to the fractional Lane-Emden equation in the case $s = 1/2$. A similar technique as in [9] allows us to show that the condition (1.6) is optimal. Indeed we have:

Theorem 1.2. *Assume $p > p_S(n)$ and that (1.6) fails. Then there are radial smooth solutions $u > 0$ with $u(r) \to 0$ as $r \to \infty$ of (1.3) that are stable.*

It is by now standard knowledge that the fractional laplacian can be seen as a Dirichlet-to-Neumann operator for a degenerate but *local* diffusion operator in the higher-dimensional half-space $\mathbb{R}^{n+1}_+$:

Theorem 1.3 ([5, 25, 28])**.** *Take $s \in (0,1)$, $\sigma > s$ and $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1 + |y|)^{n+2s}dy)$. For $X = (x,t) \in \mathbb{R}^{n+1}_+$, let*

$$\bar{u}(X) = \int_{\mathbb{R}^n} P(X,y)u(y)\,dy,$$

where

$$P(X,y) = p_{n,s}\,t^{2s}|X-y|^{-(n+2s)}$$

and $p_{n,s}$ is chosen so that $\int_{\mathbb{R}^n} P(X,y)\,dy = 1$. Then, $\bar{u} \in C^2(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+})$, $t^{1-2s}\partial_t\bar{u} \in C(\overline{\mathbb{R}^{n+1}_+})$ and

$$\begin{cases} \nabla\cdot(t^{1-2s}\nabla\bar{u}) = 0 & \text{in } \mathbb{R}^{n+1}_+,\\ \bar{u} = u & \text{on } \partial\mathbb{R}^{n+1}_+,\\ -\lim\limits_{t\to 0} t^{1-2s}\partial_t\bar{u} = \kappa_s(-\Delta)^s u & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases}$$

where

$$\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}. \tag{1.9}$$

Applying Theorem 1.3 to a solution of the fractional Lane-Emden equation, we end up with the equation

$$\begin{cases} -\nabla\cdot(t^{1-2s}\nabla\bar u)=0 & \text{in } \mathbb{R}^{n+1}_+\\ -\lim_{t\to 0} t^{1-2s}\partial_t\bar u=\kappa_s|\bar u|^{p-1}\bar u & \text{on } \partial\mathbb{R}^{n+1}_+\end{cases} \tag{1.10}$$

An essential ingredient in the proof of Theorem 1.1 is the following monotonicity formula

Theorem 1.4. *Take a solution to (1.10) $\bar u\in C^2(\mathbb{R}^{n+1}_+)\cap C(\overline{\mathbb{R}^{n+1}_+})$ such that $t^{1-2s}\partial_t\bar u\in C(\overline{\mathbb{R}^{n+1}_+})$. For $x_0\in\partial\mathbb{R}^{n+1}_+$, $\lambda>0$, let*

$$E(\bar u,x_0;\lambda)=$$

$$\lambda^{2s\frac{p+1}{p-1}-n}\left(\frac12\int_{\mathbb{R}^{n+1}_+\cap B^{n+1}(x_0,\lambda)} t^{1-2s}|\nabla\bar u|^2\,dx\,dt-\frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B^{n+1}(x_0,\lambda)}|\bar u|^{p+1}\,dx\right)$$
$$+\lambda^{2s\frac{p+1}{p-1}-n-1}\frac{s}{p+1}\int_{\partial B^{n+1}(x_0,\lambda)\cap\mathbb{R}^{n+1}_+} t^{1-2s}\bar u^2\,d\sigma.$$

Then, E is a nondecreasing function of λ. Furthermore,

$$\frac{dE}{d\lambda}=\lambda^{2s\frac{p+1}{p-1}-n+1}\int_{\partial B^{n+1}(x_0,\lambda)\cap\mathbb{R}^{n+1}_+} t^{1-2s}\left(\frac{\partial\bar u}{\partial r}+\frac{2s}{p-1}\frac{\bar u}{r}\right)^2\,d\sigma$$

Remark 2. In the above, $B^{n+1}(x_0,\lambda)$ denotes the euclidean ball in $\mathbb{R}^{n+1}$ centered at x_0 of radius λ, σ the n-dimensional Hausdorff measure restricted to $\partial B^{n+1}(x_0,\lambda)$, $r=|X|$ the euclidean norm of a point $X=(x,t)\in\ \mathbb{R}^{n+1}_+$, and $\partial_r=\nabla\cdot\frac{X}{r}$ the corresponding radial derivative.

An analogous monotonicity formula has been derived by Fall and Felli [17] to obtain unique continuation results for fractional equations. Previously, Caffarelli and Silvestre derived an Almgren quotient formula for the fractional laplacian in [5] and Caffarelli, Roquejoffre and Savin [6] obtained a related monotonicity formula to study regularity of nonlocal minimal surfaces. Another monotonicity formula for fractional problems was obtained by Cabré and Sire [3] and used by Frank, Lenzmann and Silvestre [19].

The proof of Theorem 1.1 follows an approach used in our earlier work with Kelei Wang [13] (see also [29]). First we derive suitable energy estimate (Section 2) and handle the critical and subcritical cases (Section 3). In the supercritical case, we make crucial use of a monotonicity formula Theorem 1.4, proved inSection 4. Thanks to it, using a blown-down analysis (first three steps of Section 6), we prove that the blow-down limit of a given solution is homogeneous. We then exclude the existence of stable homogeneous singular solutions in the optimal range of p (Section 5). Steps 5 and 6 of Section 6) are eventually devoted to prove that the solution itself is trivial. Finally we prove Theorem 1.2 in Section 7.

2. Energy estimates

Lemma 2.1. *Let u be a solution to (1.3). Assume that u is stable outside some ball $B^n_{R_0}\subset\mathbb{R}^n$. Let $\eta\in C^\infty_c(\mathbb{R}^n\setminus\overline{B^n_{R_0}})$ and for $x\in\mathbb{R}^n$, define*

$$\rho(x)=\int_{\mathbb{R}^n}\frac{(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}}\,dy \tag{2.1}$$

Then,

$$\int_{\mathbb{R}^n} |u|^{p+1}\eta^2\,dx + \frac{1}{p}\|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} \le \frac{\mathcal{A}_{n,s}}{p-1}\int_{\mathbb{R}^n} u^2\rho\,dx.$$

Proof. Multiply (1.3) by $u\eta^2$. Then,

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^{p+1}\eta^2\;dx &= \int_{\mathbb{R}^n} (-\Delta)^s u\, u\eta^2\;dx \\
&= \frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(u(x)\eta(x)^2-u(y)\eta(y)^2)}{|x-y|^{n+2s}}\,dx\,dy \\
&= \frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u^2(x)\eta^2(x)-u(x)u(y)(\eta^2(x)+\eta^2(y))+u^2(y)\eta^2(y)}{|x-y|^{n+2s}}\,dx\,dy \\
&= \frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)\eta(x)-u(y)\eta(y))^2-(\eta(x)-\eta(y))^2u(x)u(y)}{|x-y|^{n+2s}}\,dx\,dy \\
&= \|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} - \frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\eta(x)-\eta(y))^2u(x)u(y)}{|x-y|^{n+2s}}\,dx\,dy
\end{aligned}$$

Using the inequality $2ab \le a^2+b^2$, we deduce that

$$\|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} - \int_{\mathbb{R}^n} |u|^{p+1}\eta^2\;dx \le \frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n} u^2\rho\;dx \tag{2.2}$$

Since u is stable, we deduce that

$$(p-1)\int_{\mathbb{R}^n} |u|^{p+1}\eta^2\;dx \le \frac{\mathcal{A}_{n,s}}{2}\int_{\mathbb{R}^n} u^2\rho\;dx$$

Going back to (2.2), it follows that

$$\frac{1}{p}\|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |u|^{p+1}\eta^2\;dx \le \frac{\mathcal{A}_{n,s}}{p-1}\int_{\mathbb{R}^n} u^2\rho\;dx$$

□

Lemma 2.2. *For $m > n/2$ and $x \in \mathbb{R}^n$, let*

$$\eta(x) = (1+|x|^2)^{-m/2} \qquad \text{and} \quad \rho(x) = \int_{\mathbb{R}^n} \frac{(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}}\,dy \tag{2.3}$$

Then, there exists a constant $C = C(n,s,m) > 0$ such that

$$C^{-1}\left(1+|x|^2\right)^{-\frac{n}{2}-s} \le \rho(x) \le C\left(1+|x|^2\right)^{-\frac{n}{2}-s}. \tag{2.4}$$

Proof. Let us prove the upper bound first. Since ρ is a continuous function, we may always assume that $|x| \ge 1$. Split the integral

$$\int_{\mathbb{R}^n} \frac{(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}}\,dy$$

in the regions $|x-y| < |x|/2$, $|x|/2 \le |x-y| \le 2|x|$, and $|x-y| > 2|x|$. When $|x-y| \le |x|/2$,

$$|\eta(x)-\eta(y)| \le C(1+|x|^2)^{-\frac{m+1}{2}}|x-y|.$$

So,

$$\begin{aligned}
\int_{|x-y|\le|x|/2} \frac{(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}}\,dy &\le C(1+|x|^2)^{-m-1}\int_{|x-y|\le|x|/2} |x-y|^{2-n-2s}\,dy \\
&\le C(1+|x|^2)^{-m-s} \le C\left(1+|x|^2\right)^{-\frac{n}{2}-s}.
\end{aligned}$$

When $|x|/2 \le |x-y| \le 2|x|$,

$$\int_{|x|/2\le|x-y|\le 2|x|} \frac{(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}}\,dy \le C|x|^{-n-2s}\int_{|y|\le 2|x|}(\eta(x)^2+\eta(y)^2)\,dy$$
$$\le C|x|^{-n-2s}(|x|^{-2m+n}+1) \le C(1+|x|^2)^{-\frac{n}{2}-s},$$

where we used the assumption $m > \frac{n}{2}$. When $|x-y| > 2|x|$, then $|y| \ge |x|$ and $\eta(y) \le C(1+|x|^2)^{-m/2}$. Then,

$$\int_{|x-y|>2|x|} \frac{(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}}\,dy \le C(1+|x|^2)^{-m}\int_{|x-y|>2|x|}\frac{1}{|x-y|^{n+2s}}\,dy$$
$$\le C(1+|x|^2)^{-m-s} \le C(1+|x|^2)^{-\frac{n}{2}-s}.$$

Let us turn to the lower bound. Again, we may always assume that $|x| \ge 1$. Then,

$$\rho(x) \ge \int_{|y|\le 1/2}\frac{(\eta(y)-\eta(x))^2}{|x-y|^{n+2s}}\,dy \ge \left(\frac{|x|}{2}\right)^{-(n+2s)}\int_{|y|\le 1/2}(\eta(y)-2^{-m/2})^2\,dy$$

and the estimate follows. □

Corollary 2.3. *Let $m > n/2$, η given by (2.3), $R \ge R_0 \ge 1$, $\psi \in C^\infty(\mathbb{R}^n)$ be such that $0 \le \psi \le 1$, $\psi \equiv 0$ on B_1^n and $\psi \equiv 1$ on $\mathbb{R}^n \setminus B_2^n$. Let*

$$\eta_R(x) = \eta\left(\frac{x}{R}\right)\psi\left(\frac{x}{R_0}\right) \qquad \text{and} \quad \rho_R(x) = \int_{\mathbb{R}^n}\frac{(\eta_R(x)-\eta_R(y))^2}{|x-y|^{n+2s}}\,dy \tag{2.5}$$

There exists a constant $C = C(n,s,m,R_0) > 0$ such that for all $|x| \ge 3R_0$

$$\rho_R(x) \le C\eta\left(\frac{x}{R}\right)^2|x|^{-(n+2s)} + R^{-2s}\rho\left(\frac{x}{R}\right)$$

Proof. Fix x such that $|x| \ge 3R_0$. Using the definition of η_R and Young's inequality, we have

$$\frac{1}{2}\rho_R(x) \le \eta\left(\frac{x}{R}\right)^2\int_{\mathbb{R}^n}\frac{\left(\psi\left(\frac{x}{R_0}\right)-\psi\left(\frac{y}{R_0}\right)\right)^2}{|x-y|^{n+2s}}\,dy + \int_{\mathbb{R}^n}\psi\left(\frac{y}{R_0}\right)^2\frac{(\eta\left(\frac{x}{R}\right)-\eta\left(\frac{y}{R}\right))^2}{|x-y|^{n+2s}}\,dy$$
$$\le \eta\left(\frac{x}{R}\right)^2\int_{B^n_{2R_0}}\frac{1}{|x-y|^{n+2s}}\,dy + \int_{\mathbb{R}^n}\frac{(\eta\left(\frac{x}{R}\right)-\eta\left(\frac{y}{R}\right))^2}{|x-y|^{n+2s}}\,dy$$
$$\le C\eta\left(\frac{x}{R}\right)^2|x|^{-(n+2s)} + R^{-2s}\rho\left(\frac{x}{R}\right)$$

and the result follows. □

Lemma 2.4. *Let u be a solution of (1.3) which is stable outside a ball $B^n_{R_0}$. Take ρ_R as in Corollary 2.3 with $m \in (\frac{n}{2}, \frac{n}{2}+\frac{s(p+1)}{2})$. Then, there exists a constant $C = C(n,s,m,p,R_0) > 0$ such that for all $R \ge 3R_0$,*

$$\int_{\mathbb{R}^n}u^2\rho_R\,dx \le C\left(\int_{B^n_{3R_0}}u^2\rho_R\,dx + R^{n-2s\frac{p+1}{p-1}}\right).$$

Proof. By Corollary 2.3, if $R \ge |x| \ge 3R_0$, then

$$\rho_R(x) \le C(|x|^{-n-2s} + R^{-2s})$$

and so

$$\int_{B_R^n\setminus B_{3R_0}^n} \rho_R(x)^{\frac{p+1}{p-1}}\eta_R(x)^{-\frac{4}{p-1}}\ dx \le CR^{n-2s\frac{p+1}{p-1}}.$$

Similarly, if $|x|\ge R\ge 3R_0$, then

$$\rho_R(x)\le CR^{-2s}\left(1+\frac{|x|^2}{R^2}\right)^{-\frac{n}{2}-s}$$

and so

$$\rho_R(x)^{\frac{p+1}{p-1}}\eta_R(x)^{-\frac{4}{p-1}}\le CR^{-2s\frac{p+1}{p-1}}\left(1+\frac{|x|^2}{R^2}\right)^{-\frac{n+2s}{2}\frac{p+1}{p-1}+\frac{2m}{p-1}}$$

Since $m\in(\frac{n}{2},\frac{n}{2}+s\frac{p+1}{2})$, we have $\frac{2m}{p-1}-\frac{n+2s}{2}\frac{p+1}{p-1}<-\frac{n}{2}$ and so

$$\int_{\mathbb{R}^n\setminus B_{3R_0}^n} \rho_R(x)^{\frac{p+1}{p-1}}\eta_R(x)^{-\frac{4}{p-1}}\ dx \le CR^{n-2s\frac{p+1}{p-1}}.$$

Now,

$$\begin{aligned}\int_{\mathbb{R}^n} u^2\rho_R\ dx &= \int_{B_{3R_0}^n} u^2\rho_R\ dx + \int_{\mathbb{R}^n\setminus B_{3R_0}^n} u^2\rho_R\,\eta_R^{-\frac{4}{p+1}}\,\eta_R^{\frac{4}{p+1}}\ dx\\
&\le \int_{B_{3R_0}^n} u^2\rho_R\ dx + \left(\int_{\mathbb{R}^n}|u|^{p+1}\eta_R^2\ dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^n}\rho_R^{\frac{p+1}{p-1}}\eta_R^{-\frac{4}{p-1}}\ dx\right)^{\frac{p-1}{p+1}}\\
&\le \int_{B_{3R_0}^n} u^2\rho_R\ dx + CR^{(n-2s\frac{p+1}{p-1})\frac{p-1}{p+1}}\left(\int_{\mathbb{R}^n}|u|^{p+1}\eta_R^2\ dx\right)^{\frac{2}{p+1}}\end{aligned}$$

By a standard approximation argument, Lemma 2.1 remains valid with $\eta=\eta_R$ and $\rho=\rho_R$ and so the result follows. □

Lemma 2.5. *Assume that $p\ne\frac{n+2s}{n-2s}$. Let u be a solution to (1.3) which is stable outside a ball $B_{R_0}^n$ and $\bar u$ its extension, solving (1.10). Then, there exists a constant $C=C(n,p,s,R_0,u)>0$ such that*

$$\int_{B_R^{n+1}} t^{1-2s}\bar u^2\ dxdt\le CR^{n+2(1-s)-\frac{4s}{p-1}}$$

for any $R>3R_0$.

Proof. According to Theorem 1.3,

$$\bar u(x,t)=p_{n,s}\int_{\mathbb{R}^n}u(z)\frac{t^{2s}}{(|x-z|^2+t^2)^{\frac{n+2s}{2}}}\ dz$$

so that

$$\bar u(x,t)^2\le p_{n,s}\int_{\mathbb{R}^n}u(z)^2\frac{t^{2s}}{(|x-z|^2+t^2)^{\frac{n+2s}{2}}}\ dz.$$

So,

$$\begin{aligned}\int_{B_R^{n+1}} t^{1-2s}\bar u^2\ dxdt &\le p_{n,s}\int_{|x|\le R,z\in\mathbb{R}^n}u(z)^2\left(\int_0^R\frac{t}{(|x-z|^2+t^2)^{\frac{n+2s}{2}}}\ dt\right)dzdx\\
&\le C\int_{|x|\le R,z\in\mathbb{R}^n}u^2(z)\left\{\left(|x-z|^2\right)^{-\frac{n}{2}+1-s}-\left(|x-z|^2+R^2\right)^{-\frac{n}{2}+1-s}\right\}\ dzdx\end{aligned}$$

Split this last integral according to $|x-z|<2R$ or $|x-z|\geq 2R$. Then,

$$\int_{|x|\leq R,|x-z|<2R} u^2(z)\left\{\left(|x-z|^2\right)^{-\frac{n}{2}+1-s}-\left(|x-z|^2+R^2\right)^{-\frac{n}{2}+1-s}\right\}\,dzdx\leq$$
$$\int_{|x|\leq R,|x-z|<2R} u^2(z)\left(|x-z|^2\right)^{-\frac{n}{2}+1-s}\,dzdx\leq CR^{2(1-s)}\int_{B^{n+1}_{3R}} u^2(z)\,dz\leq$$
$$CR^{2(1-s)}\left(\int|u|^{p+1}\eta_R^2\right)^{\frac{2}{p+1}}\left(\int_{B^{n+1}_{3R}}\eta_R^{-\frac{4}{p-1}}\right)^{\frac{p-1}{p+1}}\leq$$
$$CR^{2(1-s)+n\frac{p-1}{p+1}}\left(\int u^2(z)\rho_R(z)\,dz\right)^{\frac{2}{p+1}}\leq CR^{n+2(1-s)-\frac{4s}{p-1}},$$

where we used Hölder'sin equality, then Lemma 2.1 and then Lemma 2.4. For the region $|x-z|\geq 2R$, the mean-value inequality implies that

$$\int_{|x|\leq R,|x-z|\geq 2R} u^2(z)\left\{\left(|x-z|^2\right)^{-\frac{n}{2}+1-s}-\left(|x-z|^2+R^2\right)^{-\frac{n}{2}+1-s}\right\}\,dzdx\leq$$
$$CR^2\int_{|x|\leq R,|x-z|\geq 2R} u^2(z)|x-z|^{-(n+2s)}\,dzdx\leq CR^{n+2}\int_{|z|\geq R} u^2(z)|z|^{-(n+2s)}\,dz$$
$$\leq CR^2\int_{|z|\geq R} u^2\rho\,dz\leq CR^{n+2(1-s)-\frac{4s}{p-1}},$$

where we used again Corollary 2.3 in the penultimate inequality and Lemma 2.4 in the last one. □

Lemma 2.6. *Let u be a solution to (1.3) which is stable outside a ball $B^n_{R_0}$ and $\bar u$ its extension, solving (1.10). Then, there exists a constant $C=C(n,p,s,u)>0$ such that*

$$\int_{B^{n+1}_R\cap\mathbb{R}^{n+1}_+} t^{1-2s}|\nabla\bar u|^2\,dx\,dt+\int_{B^{n+1}_R\cap\partial\mathbb{R}^{n+1}_+}|u|^{p+1}\,dx\leq CR^{n-2s\frac{p+1}{p-1}}$$

Proof. The L^{p+1} estimate follows from Lemmata 2.1 and 2.4. Now take a cut-off function $\eta\in C^1_c(\overline{\mathbb{R}^{n+1}_+})$ such that $\eta=1$ on $\mathbb{R}^{n+1}_+\cap(B^{n+1}_R\setminus B^{n+1}_{2R_0})$ and $\eta=0$ on $B^{n+1}_{R_0}\cup(\mathbb{R}^{n+1}_+\setminus B^{n+1}_{2R})$, and multiply equation (1.10) by $\bar u\eta^2$. Then,

$$\begin{aligned}\kappa_s\int_{\partial\mathbb{R}^{n+1}_+}|\bar u|^{p+1}\eta^2\,dx&=\int_{\mathbb{R}^{n+1}_+}t^{1-2s}\left\{\nabla\bar u\cdot\nabla(\bar u\eta^2)\right\}\,dx\,dt\\&=\int_{\mathbb{R}^{n+1}_+}t^{1-2s}\left\{|\nabla(\bar u\eta)|^2-\bar u^2|\nabla\eta|^2\right\}\,dx\,dt.\end{aligned}\tag{2.6}$$

Since u is stable outside $B^{n+1}_{R_0}$, so is $\bar u$ and we deduce that

$$\frac{1}{p}\int_{\mathbb{R}^{n+1}_+}t^{1-2s}|\nabla(\bar u\eta)|^2\,dx\,dt\geq\int_{\mathbb{R}^{n+1}_+}t^{1-2s}\left\{|\nabla(\bar u\eta)|^2-\bar u^2|\nabla\eta|^2\right\}\,dx\,dt.$$

In other words,

$$p'\int_{\mathbb{R}^{n+1}_+}t^{1-2s}\bar u^2|\nabla\eta|^2\,dx\,dt\geq\int_{\mathbb{R}^{n+1}_+}t^{1-2s}|\nabla(\bar u\eta)|^2\,dx\,dt,\tag{2.7}$$

where $\frac{1}{p'}+\frac{1}{p}=1$. We then apply Lemma 2.5. □

3. The subcritical case

In this section, we prove Theorem 1.1 for $1 < p \leq p_S(n)$.

Proof. Take a solution u which is stable outside some ball $B^n_{R_0}$. Apply Lemma 2.4 and let $R \to +\infty$. Since $p \leq p_S(n)$, we deduce that $u \in \dot{H}^s(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$. Multiplying the equation (1.3) by u and integrating, we deduce that

$$\int_{\mathbb{R}^n} |u|^{p+1} = \|u\|^2_{\dot{H}^s(\mathbb{R}^n)}, \tag{3.1}$$

while multiplying by u^λ given for $\lambda > 0$ and $x \in \mathbb{R}^n$ by

$$u^\lambda(x) = u(\lambda x)$$

yields

$$\int_{\mathbb{R}^n} |u|^{p-1}u^\lambda = \int_{\mathbb{R}^n} (-\Delta)^{s/2}u(-\Delta)^{s/2}u^\lambda = \lambda^s \int_{\mathbb{R}^n} ww^\lambda,$$

where $w = (-\Delta)^{s/2}u$. Following Ros-Oton and Serra [27], we use the change of variable $y = \sqrt{\lambda}\,x$ to deduce that

$$\lambda^s \int_{\mathbb{R}^n} ww^\lambda \, dx = \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w^{\sqrt{\lambda}} w^{1/\sqrt{\lambda}} \, dy$$

Hence,

$$\begin{aligned}
-\frac{n}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} &= \int_{\mathbb{R}^n} x \cdot \nabla \frac{|u|^{p+1}}{p+1} = \int_{\mathbb{R}^n} (|u|^{p-1}u)x \cdot \nabla u = \\
&\left.\frac{d}{d\lambda}\right|_{\lambda=1} \int_{\mathbb{R}^n} |u|^{p-1}uu^\lambda = \left.\frac{d}{d\lambda}\right|_{\lambda=1} \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w^{\sqrt{\lambda}} w^{1/\sqrt{\lambda}} \, dy = \\
&\quad \frac{2s-n}{2}\int_{\mathbb{R}^n} w^2 + \left.\frac{d}{d\lambda}\right|_{\lambda=1} \int_{\mathbb{R}^n} w^{\sqrt{\lambda}} w^{1/\sqrt{\lambda}} \, dy = \frac{2s-n}{2}\|u\|^2_{\dot{H}^s(\mathbb{R}^n)}
\end{aligned}$$

In the last equality, we have used the fact that $w \in C^1(\mathbb{R}^n)$, as follows by elliptic regularity. We have just proved the following Pohozaev identity

$$\frac{n}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} = \frac{n-2s}{2}\|u\|^2_{\dot{H}^s(\mathbb{R}^n)}$$

For $p < p_S(n)$, the above identity together with (3.1) force $u \equiv 0$. For $p = p_S(n)$, we are left with proving that there is no stable nontrivial solution. Since $u \in \dot{H}^s(\mathbb{R}^n)$, we may apply the stability inequlatiy (1.5) with test function $\varphi = u$, so that

$$p\int_{\mathbb{R}^n} |u|^{p+1} \leq \|u\|^2_{\dot{H}^s(\mathbb{R}^n)}.$$

This contradicts (3.1) unless $u \equiv 0$. □

In the following sections, we present several tools to study the supercritical case.

4. The monotonicity formula

In this section, we prove Theorem 1.4.

Proof. Since the equation is invariant under translation, it suffices to consider the case where the center of the considered ball is the origin $x_0 = 0$. Let

$$E_1(\bar u;\lambda) = \lambda^{2s\frac{p+1}{p-1}-n}\left(\int_{\mathbb{R}^{n+1}_+\cap B^{n+1}_\lambda} t^{1-2s}\frac{|\nabla \bar u|^2}{2}dx\,dt - \int_{\partial\mathbb{R}^{n+1}_+\cap B^{n+1}_\lambda}\frac{\kappa_s}{p+1}|\bar u|^{p+1}dx\right) \tag{4.1}$$

For $X\in\mathbb{R}^{n+1}_+$, let also

$$U(X;\lambda) = \lambda^{\frac{2s}{p-1}}\bar u(\lambda X). \tag{4.2}$$

Then, U satisfies the three following properties: U solves (1.10),

$$E_1(\bar u;\lambda) = E_1(U;1), \tag{4.3}$$

and, using subscripts to denote partial derivatives,

$$\lambda U_\lambda = \frac{2s}{p-1}U + rU_r. \tag{4.4}$$

Differentiating the right-hand side of (4.3), we find

$$\frac{dE_1}{d\lambda}(\bar u;\lambda) = \int_{\mathbb{R}^{n+1}_+\cap B^{n+1}_1} t^{1-2s}\nabla U\cdot\nabla U_\lambda\,dx\,dt - \kappa_s\int_{\partial\mathbb{R}^{n+1}_+\cap B^{n+1}_1}|U|^{p-1}U_\lambda\,dx.$$

Integrating by parts and then using (4.4),

$$\begin{aligned}\frac{dE_1}{d\lambda}(\bar u;\lambda) &= \int_{\partial B^{n+1}_1\cap\mathbb{R}^{n+1}_+} t^{1-2s}U_rU_\lambda\,d\sigma\\ &= \lambda\int_{\partial B^{n+1}_1\cap\mathbb{R}^{n+1}_+} t^{1-2s}U_\lambda^2\,d\sigma - \frac{2s}{p-1}\int_{\partial B^{n+1}_1\cap\mathbb{R}^{n+1}_+} t^{1-2s}UU_\lambda\,d\sigma\\ &= \lambda\int_{\partial B^{n+1}_1\cap\mathbb{R}^{n+1}_+} t^{1-2s}U_\lambda^2\,d\sigma - \frac{s}{p-1}\left(\int_{\partial B^{n+1}_1\cap\mathbb{R}^{n+1}_+} t^{1-2s}U^2\,d\sigma\right)_\lambda\end{aligned}$$

Scaling back, the theorem follows. ∎

5. Homogeneous solutions

Theorem 5.1. *Let $\bar u$ be a stable homogeneous solution of (1.10). Assume that $p>\frac{n+2s}{n-2s}$ and*

$$p\frac{\Gamma(\frac{n}{2}-\frac{s}{p-1})\Gamma(s+\frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2}-\frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}. \tag{5.1}$$

Then, $\bar u\equiv 0$.

Proof. Take standard polar coordinates in $\mathbb{R}^{n+1}_+$: $X=(x,t)=r\theta$, where $r=|X|$ and $\theta=\frac{X}{|X|}$. Let $\theta_1=\frac{t}{|X|}$ denote the component of θ in the t direction and $S^n_+=\{X\in\mathbb{R}^{n+1}_+ : r=1,\,\theta_1>0\}$ denote the upper unit half-sphere.

Step 1. Let $\bar u$ be a homogeneous solution of (1.10) i.e. assume that for some $\psi\in C^2(S^n_+)$,

$$\bar u(X) = r^{-\frac{2s}{p-1}}\psi(\theta).$$

Then,

$$\int_{S^n_+} \theta_1^{1-2s}|\nabla\psi|^2 + \beta \int_{S^n_+} \theta_1^{1-2s}\psi^2 = \kappa_s \int_{\partial S^n_+} |\psi|^{p+1}, \tag{5.2}$$

where κ_s is given by (1.9) and

$$\beta = \frac{2s}{p-1}\left(n - 2s - \frac{2s}{p-1}\right).$$

Indeed, since $\bar{u}$ solves (1.10) and is homogeneous, ψ solves

$$\begin{cases} -\operatorname{div}(\theta_1^{1-2s}\nabla\psi) + \beta\theta_1^{1-2s}\psi = 0 & \text{on } S^n_+ \\ -\lim_{\theta_1\to 0} \theta_1^{1-2s}\partial_{\theta_1}\psi = \kappa_s|\psi|^{p-1}\psi & \text{on } \partial S^n_+, \end{cases} \tag{5.3}$$

Multiplying (5.3) by ψ and integrating, (5.2) follows.
Step 2. For all $\varphi \in C^1(S^n_+)$,

$$\kappa_s p \int_{\partial S^n_+} |\psi|^{p-1}\varphi^2 \le \int_{S^n_+} \theta_1^{1-2s}|\nabla\varphi|^2 + \left(\frac{n-2s}{2}\right)^2 \int_{S^n_+} \theta_1^{1-2s}\varphi^2 \tag{5.4}$$

By definition, $\bar{u}$ is stable if for all $\phi \in C^1_c(\overline{\mathbb{R}^{n+1}_+})$,

$$\kappa_s p \int_{\partial\mathbb{R}^{n+1}_+} |\bar{u}|^{p-1}\phi^2 \, dx \le \int_{\mathbb{R}^{n+1}_+} t^{1-2s}|\nabla\phi|^2 \, dxdt \tag{5.5}$$

Choose a standard cut-off function $\eta_\epsilon \in C^1_c(\mathbb{R}^*_+)$ at the origin and at infinity i.e. $\chi_{(\epsilon,1/\epsilon)}(r) \le \eta_\epsilon(r) \le \chi_{(\epsilon/2,2/\epsilon)}(r)$. Let also $\varphi \in C^1(S^n_+)$, apply (5.5) with

$$\phi(X) = r^{-\frac{n-2s}{2}}\eta_\epsilon(r)\varphi(\theta) \qquad \text{for } X \in \mathbb{R}^{n+1}_+,$$

and let $\epsilon \to 0$. Inequality (5.4) follows.
Step 3. For $\alpha \in (0, \frac{n-2s}{2})$, $x \in \mathbb{R}^n \setminus \{0\}$, let

$$v_\alpha(x) = |x|^{-\frac{n-2s}{2}+\alpha}$$

and $\bar{v}_\alpha$ its extension, as defined in Theorem 1.3. Then, $\bar{v}_\alpha$ is homogeneous i.e. there exists $\phi_\alpha \in C^2(S^n_+)$ such that for $X \in \mathbb{R}^{n+1}_+ \setminus \{0\}$,

$$\bar{v}_\alpha(X) = r^{-\frac{n-2s}{2}+\alpha}\phi_\alpha(\theta).$$

In addition, for all $\varphi \in C^1(S^n_+)$,

$$\begin{aligned} \int_{S^n_+} \theta_1^{1-2s}|\nabla\varphi|^2 + \left(\left(\frac{n-2s}{2}\right)^2 - \alpha^2\right) \int_{S^n_+} \theta_1^{1-2s}\varphi^2 \\ = \kappa_s\lambda(\alpha) \int_{\partial S^n_+} \varphi^2 + \int_{S^n_+} \theta_1^{1-2s}\phi_\alpha^2 \left|\nabla\left(\frac{\varphi}{\phi_\alpha}\right)\right|^2 \end{aligned} \tag{5.6}$$

Indeed, according to Fall [16, Lemma 3.1], $\bar{v}_\alpha$ is homogeneous. Using the calculus identity stated by Fall-Felli in [17, Lemma 2.1], we get

$$\begin{cases} -\operatorname{div}(\theta_1^{1-2s}\nabla\phi_\alpha) + \left(\left(\frac{n-2s}{2}\right)^2 - \alpha^2\right)\theta_1^{1-2s}\phi_\alpha = 0 & \text{on } S^n_+ \\ \phi_\alpha = 1 & \text{on } \partial S^n_+. \end{cases} \tag{5.7}$$

Multiply equation (5.7) by φ^2/ϕ_α, integrate by parts, apply the calculus identity

$$\nabla\phi_\alpha \cdot \nabla\frac{\varphi^2}{\phi_\alpha} = |\nabla\varphi|^2 - \left|\nabla\frac{\varphi}{\phi_\alpha}\right|^2 \phi_\alpha^2$$

and recall from Fall [16, Lemma 3.1] that

$$-\lim_{t\to 0} t^{1-2s}\partial_t \overline{v}_\alpha = \kappa_s \lambda(\alpha)|x|^{-\frac{n-2s}{2}+\alpha-2s},$$

where $\lambda(\alpha)$ is given by (1.8).

Step 4. For $\alpha \in (0, \frac{n-2s}{2})$

$$\phi_0 \le \phi_\alpha \quad \text{on } S^n_+. \tag{5.8}$$

Indeed, on S^n_+,

$$\operatorname{div}(\theta_1^{1-2s}\nabla\phi_0) = \left(\frac{n-2s}{2}\right)^2 \theta_1^{1-2s}\phi_0 \ge \left(\left(\frac{n-2s}{2}\right)^2 - \alpha^2\right)\theta_1^{1-2s}\phi_0$$

so ϕ_0 is a sub-solution of (5.7). By the maximum principle, the conclusion follows.

Step 5. End of proof. Fix $\alpha \in (0, \frac{n-2s}{2})$ given by

$$\alpha = \frac{n-2s}{2} - \frac{2s}{p-1}$$

so that

$$\left(\frac{n-2s}{2}\right)^2 - \alpha^2 = \frac{2s}{p-1}\left(n-2s-\frac{2s}{p-1}\right) = \beta,$$

where β is the constant appearing in (5.3).

Use the stability inequality (5.4) with $\varphi = \frac{\psi\phi_0}{\phi_\alpha}$:

$$\kappa_s p\int_{\partial S^n_+} |\psi|^{p+1} \le \int_{S^n_+} \theta_1^{1-2s}\left|\nabla\left(\frac{\psi\phi_0}{\phi_\alpha}\right)\right|^2 + \left(\frac{n-2s}{2}\right)^2\int_{S^n_+}\theta_1^{1-2s}\left(\frac{\psi\phi_0}{\phi_\alpha}\right)^2. \tag{5.9}$$

Note that a particular case of the identity (5.6) is

(5.10)
$$\int_{S^n_+}\theta_1^{1-2s}|\nabla\varphi|^2 + \left(\frac{n-2s}{2}\right)^2\int_{S^n_+}\theta_1^{1-2s}\varphi^2 = \kappa_s\Lambda_{n,s}\int_{\partial S^n_+}\varphi^2 + \int_{S^n_+}\theta_1^{1-2s}\phi_0^2\left|\nabla\left(\frac{\varphi}{\phi_0}\right)\right|^2$$

Using (5.10) (with $\varphi = \frac{\psi\phi_0}{\phi_\alpha}$), (5.9) becomes

$$\kappa_s p\int_{\partial S^n_+}|\psi|^{p+1} \le \kappa_s\Lambda_{n,s}\int_{\partial S^n_+}\psi^2 + \int_{S^n_+}\theta_1^{1-2s}\phi_0^2\left|\nabla\left(\frac{\psi}{\phi_\alpha}\right)\right|^2.$$

By (5.8), we deduce that

$$\kappa_s p\int_{\partial S^n_+}|\psi|^{p+1} \le \kappa_s\Lambda_{n,s}\int_{\partial S^n_+}\psi^2 + \int_{S^n_+}\theta_1^{1-2s}\phi_\alpha^2\left|\nabla\left(\frac{\psi}{\phi_\alpha}\right)\right|^2.$$

Using again the identity (5.6), we deduce that

$$\kappa_s p\int_{\partial S^n_+}|\psi|^{p+1} \le \kappa_s(\Lambda_{n,s}-\lambda(\alpha))\int_{\partial S^n_+}\psi^2 + \int_{S^n_+}\theta_1^{1-2s}|\nabla\psi|^2 + \beta\int_{S^n_+}\theta_1^{1-2s}\psi^2$$

Comparing with (5.2), it follows that

$$(p-1)\int_{\partial S^n_+}|\psi|^{p+1} \le (\Lambda_{n,s}-\lambda(\alpha))\int_{\partial S^n_+}\psi^2. \tag{5.11}$$

But from (5.2) and (5.6)

$$\int_{\partial S^n_+}|\psi|^{p+1} \ge \lambda(\alpha)\int_{\partial S^n_+}\psi^2$$

Combined with (5.11), we find that

$$\lambda(\alpha)p \le \Lambda_{n,s}$$

unless $\psi \equiv 0$. □

6. Blow-down analysis

Proof of Theorem 1.1. Assume that $p > p_S(n)$. Take a solution u of (1.3) which is stable outside the ball of radius R_0 and let $\bar u$ be its extension solving (1.10).

Step 1. $\lim_{\lambda\to+\infty} E(\bar u, 0; \lambda) < +\infty$.

Since E is nondecreasing, it suffices to show that $E(\bar u, 0; \lambda)$ is bounded. Write $E = E_1 + E_2$, where E_1 is given by (4.1) and

$$E_2(\bar u;\lambda) = \lambda^{2s\frac{p+1}{p-1}-n-1}\frac{s}{p+1}\int_{\partial B^{n+1}(0,\lambda)\cap\mathbb{R}^{n+1}_+} t^{1-2s}\bar u^2\, d\sigma$$

By Lemma 2.6, E_1 is bounded. Since E is nondecreasing,

$$E(\bar u;\lambda) \le \frac{1}{\lambda}\int_\lambda^{2\lambda} E(u;t)\,dt \le C + \lambda^{2s\frac{p+1}{p-1}-n-1}\int_{B^{n+1}_{2\lambda}\cap\mathbb{R}^{n+1}_+} t^{1-2s}\bar u^2.$$

Applying Lemma 2.5, we deduce that E is bounded.

Step 2. There exists a sequence $\lambda_i \to +\infty$ such that $(\bar u^{\lambda_i})$ converges weakly in $H^1_{loc}(\mathbb{R}^{n+1}_+; t^{1-2s}dxdt)$ to a function $\bar u^\infty$.

This follows from the fact that $(\bar u^{\lambda_i})$ is bounded in $H^1_{loc}(\mathbb{R}^{n+1}_+; t^{1-2s}dxdt)$ by Lemma 2.6.

Step 3. $\bar u^\infty$ is homogeneous

To see this, apply the scale invariance of E, its finiteness and the monotonicity formula: given $R_2 > R_1 > 0$,

$$\begin{aligned}
0 &= \lim_{i\to+\infty} E(\bar u;\lambda_i R_2) - E(\bar u;\lambda_i R_1)\\
&= \lim_{i\to+\infty} E(\bar u^{\lambda_i};R_2) - E(\bar u^{\lambda_i};R_1)\\
&\ge \liminf_{i\to+\infty}\int_{(B^{n+1}_{R_2}\setminus B^{n+1}_{R_1})\cap\mathbb{R}^{n+1}_+} t^{1-2s}r^{2-n+\frac{4s}{p-1}}\left(\frac{2s}{p-1}\frac{\bar u^{\lambda_i}}{r}+\frac{\partial \bar u^{\lambda_i}}{\partial r}\right)^2 dx\,dt\\
&\ge \int_{(B^{n+1}_{R_2}\setminus B^{n+1}_{R_1})\cap\mathbb{R}^{n+1}_+} t^{1-2s}r^{2-n+\frac{4s}{p-1}}\left(\frac{2s}{p-1}\frac{\bar u^{\infty}}{r}+\frac{\partial \bar u^{\infty}}{\partial r}\right)^2 dx\,dt
\end{aligned}$$

Note that in the last inequality we only used the weak convergence of $(\bar u^{\lambda_i})$ to $\bar u^\infty$ in $H^1_{loc}(\mathbb{R}^{n+1}_+; t^{1-2s}dxdt)$. So,

$$\frac{2s}{p-1}\frac{\bar u^\infty}{r}+\frac{\partial \bar u^\infty}{\partial r} = 0 \quad a.e. \text{ in } \mathbb{R}^{n+1}_+.$$

And so, u^∞ is homogeneous.

Step 4. $\bar u^\infty \equiv 0$

Simply apply Theorem 5.1.

Step 5. $(\bar u^{\lambda_i})$ converges strongly to zero in $H^1(B_R^{n+1}\setminus B_\varepsilon^{n+1}; t^{1-2s}dxdt)$ and (u^{λ_i}) converges strongly to zero in $L^{p+1}(B_R^{n+1}\setminus B_\varepsilon^{n+1})$ for all $R>\epsilon>0$. Indeed, by Steps 2 and 3, $(\bar u^{\lambda_i})$ is bounded in $H^1_{loc}(\mathbb{R}^{n+1}_+; t^{1-2s}dxdt)$ and converges weakly to 0. It follows that $(\bar u^{\lambda_i})$ converges strongly to 0 in $L^2_{loc}(\mathbb{R}^{n+1}_+; t^{1-2s}dxdt)$. Indeed, by the standard Rellich-Kondrachov theorem and a diagonal argument, passing to a subsequence we obtain

$$\int_{\mathbb{R}^{n+1}_+\cap(B_R^{n+1}\setminus A)} t^{1-2s}|\bar u^{\lambda_i}|^2\,dxdt\to 0,$$

as $i\to\infty$, for any $B_R^{n+1}=B_R^{n+1}(0)\subset\mathbb{R}^{n+1}$ and A of the form $A=\{(x,t)\in\mathbb{R}^{n+1}_+ : 0<t<r/2\}$, where $R,r>0$. By [15, Theorem 1.2],

$$\int_{\mathbb{R}^{n+1}_+\cap B_r^{n+1}(x)} t^{1-2s}|\bar u^{\lambda_i}|^2\,dxdt\le Cr^2\int_{\mathbb{R}^{n+1}_+\cap B_r^{n+1}(x)} t^{1-2s}|\nabla\bar u^{\lambda_i}|^2\,dxdt$$

for any $x\in\partial\mathbb{R}^{n+1}_+$, $|x|\le R$, with a uniform constant C. Covering $B_R^{n+1}\cap A$ with half balls $B_r^{n+1}(x)\cap\mathbb{R}^{n+1}_+$, $x\in\partial\mathbb{R}^{n+1}_+$ with finite overlap, we see that

$$\int_{B_R^{n+1}\cap A} t^{1-2s}|\bar u^{\lambda_i}|^2\,dxdt\le Cr^2\int_{B_R^{n+1}\cap A} t^{1-2s}|\nabla\bar u^{\lambda_i}|^2\,dxdt\le Cr^2,$$

and from this we conclude that $(\bar u^{\lambda_i})$ converges strongly to 0 in $L^2_{loc}(\mathbb{R}^{n+1}_+; t^{1-2s}dxdt)$.

Now, using (2.7), $(\bar u^{\lambda_i})$ converges strongly to 0 in $H^1_{loc}(\mathbb{R}^{n+1}_+\setminus\{0\}; t^{1-2s}dxdt)$ and by (2.6), the convergence also holds in $L^{p+1}_{loc}(\mathbb{R}^n\setminus\{0\})$.

Step 6. $\bar u\equiv 0$.

Indeed,

$$\begin{aligned}
E_1(\bar u;\lambda)=E_1(\bar u^\lambda;1)&=\int_{\mathbb{R}^{n+1}_+\cap B_1^{n+1}} t^{1-2s}\frac{|\nabla\bar u^\lambda|^2}{2}dx\,dt-\int_{\partial\mathbb{R}^{n+1}_+\cap B_1^{n+1}}\frac{\kappa_s}{p+1}|\bar u^\lambda|^{p+1}dx\\
&=\int_{\mathbb{R}^{n+1}_+\cap B_\epsilon^{n+1}} t^{1-2s}\frac{|\nabla\bar u^\lambda|^2}{2}dx\,dt-\int_{\partial\mathbb{R}^{n+1}_+\cap B_\epsilon^{n+1}}\frac{\kappa_s}{p+1}|\bar u^\lambda|^{p+1}dx+\\
&\int_{\mathbb{R}^{n+1}_+\cap B_1^{n+1}\setminus B_\epsilon^{n+1}} t^{1-2s}\frac{|\nabla\bar u^\lambda|^2}{2}dx\,dt-\int_{\partial\mathbb{R}^{n+1}_+\cap B_1^{n+1}\setminus B_\epsilon^{n+1}}\frac{\kappa_s}{p+1}|\bar u^\lambda|^{p+1}dx\\
&=\varepsilon^{n-2s\frac{p+1}{p-1}}E_1(\bar u;\lambda\varepsilon)+\int_{\mathbb{R}^{n+1}_+\cap B_1^{n+1}\setminus B_\epsilon^{n+1}} t^{1-2s}\frac{|\nabla\bar u^\lambda|^2}{2}dx\,dt-\int_{\partial\mathbb{R}^{n+1}_+\cap B_1^{n+1}\setminus B_\epsilon^{n+1}}\frac{\kappa_s}{p+1}|\bar u^\lambda|^{p+1}dx\\
&\le C\varepsilon^{n-2s\frac{p+1}{p-1}}+\int_{\mathbb{R}^{n+1}_+\cap B_1^{n+1}\setminus B_\epsilon^{n+1}} t^{1-2s}\frac{|\nabla\bar u^\lambda|^2}{2}dx\,dt-\int_{\partial\mathbb{R}^{n+1}_+\cap B_1^{n+1}\setminus B_\epsilon^{n+1}}\frac{\kappa_s}{p+1}|\bar u^\lambda|^{p+1}dx
\end{aligned}$$

Letting $\lambda\to+\infty$ and then $\varepsilon\to 0$, we deduce that $\lim_{\lambda\to+\infty}E_1(\bar u;\lambda)=0$. Using the monotonicity of E,

$$E(\bar u;\lambda)\le\frac{1}{\lambda}\int_\lambda^{2\lambda}E(t)\,dt\le\sup_{[\lambda,2\lambda]}E_1+C\lambda^{-n-1+2s\frac{p+1}{p-1}}\int_{B_{2\lambda}^{n+1}\setminus B_\lambda^{n+1}}\bar u^2$$

and so $\lim_{\lambda\to+\infty} E(\bar{u};\lambda) = 0$. Since u is smooth, we also have $E(\bar{u};0) = 0$. Since E is monotone, $E \equiv 0$ and so $\bar{u}$ must be homogeneous, a contradiction unless $\overline{u} \equiv 0$. $\square$

7. Construction of radial entire stable solutions

Let $\bar{u}_s$ denote the extension of the singular solution u_s (1.7) to $\mathbb{R}^{n+1}_+$ defined by

$$\bar{u}_s(X) = \int_{\mathbb{R}^n} P(X,y)u(y)\,dy.$$

Let B_1^{n+1} denote the unit ball in $\mathbb{R}^{n+1}$ and for $\lambda \geq 0$, consider

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla u) = 0 & \text{in } B_1^{n+1} \cap \mathbb{R}^{n+1}_+ \\ u = \lambda \bar{u}_s & \text{on } \partial B_1^{n+1} \cap \mathbb{R}^{n+1}_+ \\ -\lim_{t\to 0}(t^{1-2s}u_t) = \kappa_s u^p & \text{on } B_1^{n+1} \cap \{t=0\}. \end{cases} \tag{7.1}$$

Take $\lambda \in (0,1)$. Since u_s is a positive supersolution of (7.1), there exists a minimal solution $u = u_\lambda$. By minimality, the family (u_λ) is nondecreasing and u_λ is axially symmetric, that is, $u_\lambda(x,t) = u_\lambda(r,t)$ with $r = |x| \in [0,1]$. In addition, for a fixed value $\lambda \in (0,1)$, u_λ is bounded, as can be proved by the truncation method of [1], see also [10] and radially decreasing by the moving plane method (see [7] for a similar setting). From now on let us assume that $p_S(n) < p$ and

$$p\frac{\Gamma(\frac{n}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} \leq \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2},$$

which means that the singular solution u_s is stable. Then, $u_\lambda \uparrow u_s$ as $\lambda \uparrow 1$, using the classical convexity argument in [2] (see also Section 3.2.2 in [14]). Let $\lambda_j \uparrow 1$ and

$$m_j = \|u_{\lambda_j}\|_{L^\infty} = u_{\lambda_j}(0), \quad R_j = m_j^{\frac{p-1}{2s}},$$

so that m_j, $R_j \to \infty$ as $j \to \infty$. Set

$$v_j(x) = m_j^{-1} u_{\lambda_j}(x/R_j).$$

Then $0 \leq v_j \leq 1$ is a bounded solution of

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla v_j) = 0 & \text{in } B_{R_j}^{n+1} \cap \mathbb{R}^{n+1}_+ \\ v_j = \lambda_j \bar{u}_s & \text{on } \partial B_{R_j}^{n+1} \cap \mathbb{R}^{n+1}_+ \\ -\lim_{t\to 0}(t^{1-2s}(v_j)_t) = \kappa_s v_j^p & \text{on } B_{R_j}^{n+1} \cap \{t=0\}. \end{cases}$$

Moreover $v_j \leq \bar{u}_s$ in $B_{R_j}^{n+1} \cap \mathbb{R}^{n+1}_+$ and $v_j(0) = 1$. Using elliptic estimates we find (for a subsequence) that (v_j) converges uniformly on compact sets of $\overline{\mathbb{R}}^{n+1}_+$ to a function v that is axially symmetric and solves

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\lim_{t\to 0}(t^{1-2s}v_t) = \kappa_s v^p & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Moreover $0 \leq v \leq 1$, $v(0) = 1$ and $v \leq \bar{u}_s$. This v restricted to $\mathbb{R}^n \times \{0\}$ is a radial, bounded, smooth solution of (1.3) and from $v \leq \bar{u}_s$ we deduce that v is stable.

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