IFS attractors and Cantor sets

Sylvain Crovisier, Michal Rams

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Abstract

We build a metric space which is homeomorphic to a Cantor set but cannot be realized as the attractor of an iterated function system. We give also an example of a Cantor set K in $\mathbb{R}^3$ such that every homeomorphism f of $\mathbb{R}^3$ which preserves K coincides with the identity on K.

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1 Introduction

We are interested in the problem of characterization of compact sets that are limit sets of hyperbolic dynamical systems. In this paper we study iterated function systems that are simplified models for the smooth hyperbolic dynamics. Previous works (see [DH, K]) investigated which compact metric spaces can be attractors of iterated function systems on some euclidean space. We would like to carry on this discussion with the following question:

Is every Cantor set an attractor of some iterated function system?

Let us first recall that a continuous mapping f of a metric space (X, d) into itself is a *contractive map* if there exists a constant $\sigma \in (0, 1)$ such that for any points x, y in X, the distance $d(f(x), f(y))$ is less or equal to $\sigma d(x, y)$. An *iterated function system* (or *IFS*) is a finite family of contractive maps $\{f_1, \dots, f_s\}$ acting on a complete metric space. It is well known (see [H]) that such a dynamics possesses a unique compact set $K \subset X$ which is non-empty and invariant by the IFS, in the following sense:

$$K = \cup_{i=1}^{s} f_i(K).$$

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One calls this compact set the limit set or the *attractor* of the IFS.

As an example, the middle one third Cantor set is the attractor of the IFS $\{f_1, f_2\}$ on $\mathbb{R}$ defined by

$$f_1(x) = x/3, \quad f_2(x) = (x+2)/3.$$

Our first result shows that for some other metrics, the Cantor set is no more the attractor of an IFS.

Theorem 1.1. *There exists a Cantor set X_1 and a Borel probability measure μ supported on X_1 such that for every contractive map $f\colon X_1 \to X_1$, we have $\mu(f(X_1)) = 0$.*

This set X_1 cannot be an attractor of an iterated function system $\{f_1, \dots, f_s\}$ (even if one allows countably many maps in the definition of the IFS) since X_1 has full measure by μ but $f_1(X_1) \cup \dots \cup f_s(X_1)$ has zero measure by μ. The example may be generalized:

Corollary 1.2. *For any iterated function system on a complete metric space (X, d), the attractor K is not isometric to the Cantor set X_1, defined in Theorem 1.1.*

In the previous case, the obstruction was metrical. If one specifies the Cantor set K and the ambient space X there may exist also topological obstructions for K to be an attractor of an IFS defined on X, even if X is a smooth manifold such as $\mathbb{R}^d$.

Theorem 1.3. *There exists a Cantor set $X_2 \subset \mathbb{R}^3$ such that if f is a homeomorphism of $\mathbb{R}^3$ which satisfies $f(X_2) \subset X_2$ then $f_{|X_2} = \mathrm{id}$.*

In particular a finite set of homeomorphisms of $\mathbb{R}^3$ can not be an IFS whose attractor is X_2. The set X_2 here is a variation on Antoine's necklace.

This example can be easily generalized to higher dimensions but in dimension one or two the situation is completely different:

Proposition 1.4. *For any Cantor set $X \in \mathbb{R}$ (or in $\mathbb{R}^2$) and any two points $x, y \in X$, there exists a contractive homeomorphism $f : \mathbb{R} \to \mathbb{R}$ (or $f : \mathbb{R}^2 \to \mathbb{R}^2$) such that $f(X) \subset X$ and $f(x) = y$.*

2 Constructions

Given Y, subset of a metric space X we denote its complement by $(Y)^c$, its interior $\mathrm{Int}(Y)$, its boundary $\partial(Y)$ and (in the case Y is bounded) its diameter $\mathrm{diam}(Y)$. We will denote by $B(z, r)$ the open ball centered at $z \in X$ with radius r.

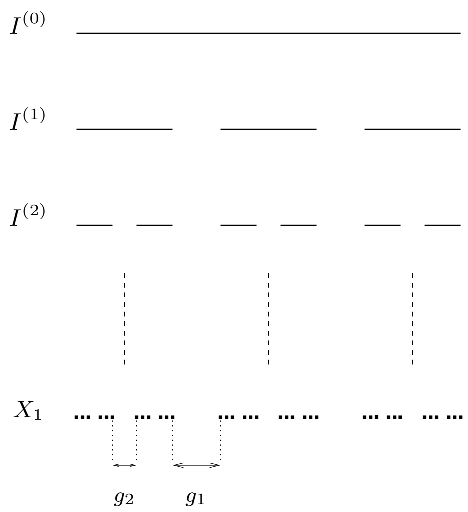

Figure 1: Construction of X_1

2.1 Proof of Theorem 1.1

2.1.1 Definition of the Cantor set X_1 and the measure μ

The Cantor set X_1 is obtained as the intersection of a decreasing sequence $(I^{(k)})$ of compact sets in $\mathbb{R}$. Each set $I^{(k)}$ is a finite union of pairwise disjoint compact intervals $I_i^{(k)}$ that have the same length.

We will construct inductively the sequence $(I^{(k)})$. The first set $I^{(0)} = [0, 1]$.

Given $I^{(k)} = \bigcup I_j^{(k)}$,we choose n_{k+1} closed intervals $I_i^{(k+1)}$ inside every $I_j^{(k)}$. We demand those intervals to be pairwise disjoint and of equal length, that their union contains the endpoints of $I_j^{(k)}$ and also that the gaps between them are of equal length g_{k+1}. The set $I^{(k+1)}$ will be the union of all $I_i^{(k+1)}$.

Obviously, the bounded components of $\mathbb{R} \setminus I^{(k+1)}$ are either gaps created at the previous steps of the construction or new gaps of size g_{k+1} each.

We take care along the induction that in each consecutive step the number of intervals n_k increases while the size of gaps g_k decreases. We define $X_1 = \bigcap_k I^{(k)}$ which is obviously a Cantor subset of $\mathbb{R}$, see the figure 1.

We define the measure μ as to be equidistributed: all the intervals $I_i^{(k)}$ of a given level k have the same measure, equal to the reciprocal of the total number of intervals of that level, ie. $\mu(I_i^{(k)}) = \prod_{j=1}^{k} n_j^{-1}$. By the Kolmogorov theorem ([B2]) it uniquely defines a probability measure supported on X_1.

2.1.2 Measure of $f(X_1)$

Now, we prove that X_1 satisfies the assertion of Theorem 1.1: let f be a contraction from X_1 into itself and let $X_i^{(k)} = X_1 \cap I_i^{(k)}$ be the part of X_1 contained in one of the k-th level intervals $I_i^{(k)}$. We claim that $f(X_i^{(k)})$ must be contained in some $(k+1)$-th level interval.

Assuming it is not the case, $f(X_i^{(k)})$ intersects at least two $(k+1)$-th level intervals. Since (g_n) is decreasing, these intervals are in distance at least g_{k+1} from each other. Hence, $f(X_i^{(k)})$ may be divided onto two subsets A, B such that the distance between any point from A and any point from B is not smaller than g_{k+1}. As the map f is contracting, the preimages of A and B (covering together whole $X_i^{(k)}$) must lie in distance strictly greater than g_{k+1}. But if we could divide $X_i^{(k)}$ into two such subsets, this would imply the existence of a gap inside $X_i^{(k)}$ of size strictly greater than g_{k+1}. By construction, such a gap would be created at a step ℓ larger or equal to $k+1$ and would be of length g_ℓ. The sequence (g_n) would not be strictly decreasing. This contradiction proves the claim.

Hence, as there are $\prod_{j=1}^{k} n_j$ intervals of level k in $I^{(k)}$, the whole set $f(X_1)$ is contained in the union of at most $\prod_{j=1}^{k} n_j$ $(k+1)$-th level intervals. This implies

$$\mu(f(X_1)) \leq \prod_{i=1}^{k} n_i \prod_{j=1}^{k+1} n_j^{-1} = n_{k+1}^{-1}$$

and the right hand side tends to zero when k tends to infinity. □

2.2 Proof of Theorem 1.3

2.2.1 Definition of the Cantor set X_2

The construction is a variation on Antoine's necklace example built in [A$_1$, A$_2$] (see also for example [M], chapter 18 or [B1], section IV.7 for more details).

We start from a filled compact torus $T^{(0)} \subset \mathbb{R}^3$ that will also be noted $T_\emptyset$. In the first step we find in the interior of $T^{(0)}$ some number (say, $n = n_\emptyset > 2$) of disjoint compact tori $T_1, \dots, T_n$ linked together to form a closed chain going around the torus $T^{(0)}$, see figure 2. They will be called first level tori. We denote their union by $T^{(1)}$. In the second step we take inside each of the first level tori T_i another chain of smaller tori (called second level tori) going around (of n_1 elements inside T_1, n_2 inside T_2 and so on up to $n_{n_\emptyset}$). They will be denoted by $T_{i,1}, \dots, T_{i,n_i}$ and their union by $T^{(2)} \subset T^{(1)}$.

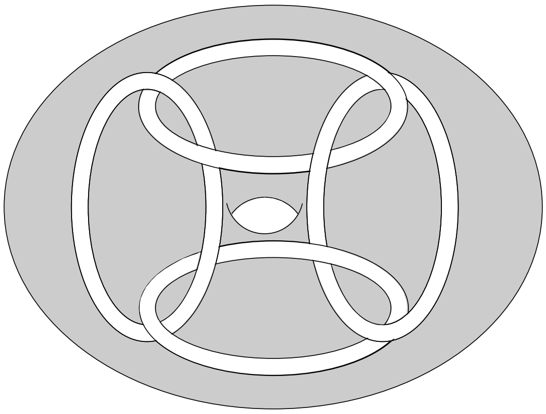

Figure 2: The first sets $T^{(0)}$ and $T^{(1)}$ (here we have $n_\emptyset = 4$)

We repeat the procedure inductively, taking care that the diameters of the tori we use in the construction go down to 0 and that, at any step, the lengths of the chains are greater than 2, different from each other, and different from the lengths of all other chains introduced at the previous steps (in Antoine's example one takes chains of length four at every step). All the tori of level k are described by finite sequences $\omega^k \in \Sigma$ of length k where:

$$\Sigma = \bigcup_{k\in\mathbb{N}} \{\omega^k = (\omega_1, \ldots, \omega_k) \in \mathbb{N}^k;\ \forall_{j\leq k}\ 1 \leq \omega_j \leq n_{\omega^{k-1}}\}.$$

(We denoted by ω^k a sequence of length k, by ω_j the j-th element of the sequence and by ω^j the subsequence formed by the j first terms in ω^k.)

We define

$$X_2 = \bigcap_k T^{(k)}.$$

As one can easily check from the construction, X_2 is a Cantor set.

We introduce also the *rings of level* k in X_2 defined by

$$Y_{\omega^k} = X_2 \cap T_{\omega^k}.$$

Let us remark that the sets Y_{ω^k} are open and closed in X_2. They define a basis for the topology on X_2.

A *chain* C in X_2 is a union of rings of same level:

$$C = Y_{\omega^k, i_1} \cup Y_{\omega^k, i_2} \cup \cdots \cup Y_{\omega^k, i_r}$$

where ω^k belongs to Σ and where $\{i_1, \ldots, i_r\}$ is an interval in $\mathbb{Z}/n_{\omega^k}\mathbb{Z}$.

2.2.2 Topological properties of X_2

Let Y be a compact set of $\mathbb{R}^3$. We will say that Y *may be cleaved* if there exists a decomposition $Y = A_1 \cup A_2$ of Y in two compact disjoint and non empty sets and some isotopy of homeomorphisms $(h_t)_{t\in[0,1]}$ of such that $h_0 = \mathrm{id}$ and such that $h_1(A_1)$ and $h_1(A_2)$ are contained in two disjoint euclidean balls of $\mathbb{R}^3$.

We prove in this section the following topological characterization of rings of X_2:

Proposition 2.1. *A compact subset Y of X_2 is a ring of X_2 if and only if it satisfies the following properties:*

a) Y can not be cleaved.

b) Y is "cyclic": there exists a partition of Y in three disjoint compact non empty subsets $Y = A_1 \cup A_2 \cup A_3$ such that each A_i and each union $A_i \cup A_j$ can not be cleaved.

We first check that rings satisfy these properties.

Lemma 2.2. *No chain C of X_2 may be cleaved.*

In particular the rings can not be cleaved.

Proof. In [A$_1$, A$_2$] (see also [M], problem 18.2), Antoine proved that any 2-sphere in $\mathbb{R}^3$ that do not intersect the Antoine's necklace can not separate two points of the necklace from one another. By the same argument, this property is also satisfied by any chain of X_2. This implies the lemma. □

We then check that rings are "cyclic" (i.e. satisfy Proposition 2.1 b)): let Y_{ω^k} be a ring of X_2. We set $A_1 = Y_{\omega^k,1}$, $A_2 = Y_{\omega^k,2}$ and $A_3 = Y_{\omega^k} \setminus (A_1 \cup A_2)$. All the sets A_i or $A_i \cup A_j$ are chains of X_2 and can not be not cleaved from lemma 2.2, what was to be shown.

The other part of the proof of Proposition 2.1 uses the following lemma.

Lemma 2.3. *Let Y be a compact and proper subset of some ring Y_{ω^k} in X_2. Then, there exists an isotopy $(h_t)_{t\in[0,1]}$ in the space of homeomorphisms such that h_t coincides with the identity on $(\mathrm{Int}(T_\omega))^c$, $h_0 = \mathrm{id}$ and h_1 sends Y on a small euclidean ball included in $\mathrm{Int}(T_\omega)$.*

Proof. This lemma is clear when Y is a chain $Y_{\omega^k,i_1}, \dots, Y_{\omega^k,i_r}$ of T_{ω^k}. In the general case, we note that $Y_{\omega^k} \setminus Y$ is open in Y_{ω^k} and contained in some ring. Hence, Y is contained in a finite union of pairwise disjoint rings that is strictly included in Y_{ω^k}. One observes that it is enough to prove the assertion for such union of rings, it will imply the result for Y. Note that such a union of rings is a finite union of pairwise disjoint chains not linked to each other.

The proof is then done by induction on the number of chains : from the remark we did at the beginning of this proof, one starts by shrinking the chains of largest order inside small disjoint balls. This allows us to consider the chains of next largest order and to shrink them. Repeating this procedure, one concludes the proof. □

We get a counterpart to lemma 2.2

Lemma 2.4. *Let Y be a closed subset of X_2. If Y can not be cleaved and contains at least two points, then it is a chain of X_2.*

Proof. Let Y be a closed subset of X_2 that contains at least two points and can not be cleaved. The proof that Y is a chain has three steps.
Step 1: *Let Y_{ω^k} be any ring of X_2, such that Y intersects both Y_{ω^k} and $(Y_{\omega^k})^c$. Then, Y_{ω^k} is contained in Y.*
We prove this fact by contradiction. If Y_{ω^k} is not contained in Y, one can apply Lemma 2.3: the set $A_1 = Y \cap Y_{\omega^k}$ may be contracted to a small euclidean ball contained in $\mathrm{Int}(T_{\omega^k})$. Since $(Y)^c$ is connected, one can then send this ball outside any bounded neighborhood U of $T^{(0)}$, by an isotopy that fixes the closed set $Y \setminus A_1$. The closed set $A_2 = Y \setminus A_1$ is not empty by assumption and it is possible to contract it in a small euclidean ball through an isotopy whose support is contained in U. This shows that Y can be cleaved. This is a contradiction and the fact is proved.
Step 2: *There exists a word ω^k such that Y is a union of some rings of the form $Y_{\omega^k,\ell}$ (with the same level $k+1$).*
Let z_1 be any point in Y. By assumption, Y contains at least an other point z_2. By construction of X_2, there exists a ring Y_{ω^k} that contains z_1 and not z_2. By the first step, Y_{ω_k} is included in Y. We hence proved that Y is a union of rings of X.

Let us note that, for any two rings A_1 and A_2 of X_2, either they are disjoint or one is contained in the other. Consequently, Y is a disjoint union of rings that are maximal in Y for the inclusion.

Let $A \subset Y$ be one of these maximal rings and let A' be the ring of X_2 which contains A and was built at the previous level. Hence, A' is not contained in Y. By the first step of this proof, one deduces that Y is contained in A'. Let us denote A' by T_{ω^k}. Repeating this argument with any maximal subring of Y, one deduces that Y is a union of rings of the form $Y_{\omega^k,i}$. This proves the second step.
Step 3: *The set Y is a chain.*
From the second step, Y decomposes as an union of chains made of rings of

the form $Y_{\omega^k,i}$. If Y is not a chain, it decomposes as chains $C_1, \dots, C_r$, that are not linked together. Hence, one can contract C_1 and $C_2 \cup \dots \cup C_r$ in two disjoint euclidean balls. This is a contradiction since Y can not be not cleaved. Hence, Y is a single chain. □

End of the proof of Proposition 2.1. Let Y be a closed subset of X_2 that satisfies the properties of Proposition 2.1.

By the second property, Y decomposes as a union $A_1 \cup A_2 \cup A_3$. By Lemma 2.4, all the sets A_i are chains of X_2. By the same lemma, the unions $A_i \cup A_j$ are chains as well, hence A_1, A_2 and A_3 are unions of rings of the form $Y_{\omega^k,i}$ contained in a same ring Y_{ω^k}. But since they are disjoint the only way that $A_1 \cup A_2$, $A_2 \cup A_3$ and $A_3 \cup A_1$ are all chains of X_2 is that their union is the ring Y_{ω^k}. This shows that Y is a ring and concludes the proof. □

2.2.3 Rigidity of X_2 under homeomorphisms

We now end the proof of Theorem 1.3.

Let f be any homeomorphism from $\mathbb{R}^3$ onto itself such that $f(X_2) \subset X_2$. From the Proposition 2.1, the image of any ring Y_{ω^k} by f is a ring Y_{τ^j}.

Lemma 2.5. *Let Y_{ω^k} be a ring of X_2 and Y_{τ^j} its image by f. The images of the subrings $Y_{\omega^k,i}$ of level $k+1$ of Y_{ω^k} are subrings $Y_{\tau^j,\ell}$ of level $j+1$ of Y_{τ^j}.*

Proof. Let us assume that it is not the case. There would exists a ring Y of level $j+1$ in Y_{τ^j} which contains strictly the image of some subring $Y_{\omega^k,i}$ of Y_{ω^k}. In other words, the preimage $f^{-1}(Y)$ is a proper subset of Y_{ω^k} but is larger than the subrings of level $k+1$ of Y_{ω^k}. However, by Proposition 2.1, the set $f^{-1}(Y)$ is a ring of X_2. This is a contradiction. □

By this lemma, the rings Y_{ω^k} and Y_{τ^j} decompose in the same number of subrings of next level so that $n_{\omega^k} = n_{\tau^j}$. By the construction, the lengths of the closed chains of tori in X_2 are all different. This implies that $Y_{\omega^k} = Y_{\tau^j}$. Since any point $z \in X_2$ is the intersection of some family of rings Y_{ω^k} (with $k \nearrow \infty$), it is fixed by the map f. Hence, the restriction of f to X_2 is the identity. This concludes the proof of Theorem 1.3.

2.3 Proof of Proposition 1.4

Let X be a Cantor set on the plane (the proof for the Cantor set on the line is similar but simpler and will be left for the reader as an exercise). Let $\Sigma = \bigcup_{n=0}^{\infty} \{0,1\}^n$ be the space of all finite zero-one sequences.

2.3.1 Preliminary constructions

We start by the construction of the covering of X with a family of topological closed balls B_{ω^n} with $\omega^n \in \Sigma$, satisfying the following properties:

i) $\forall_{i\in\{0,1\}} \quad B_{\omega^n,i} \subset \operatorname{Int} B_{\omega^n}$,

ii) $\forall_{\omega^n\in\Sigma} \quad \forall_{i\in\{0,1\}} \quad B_{\omega^n,0} \cap B_{\omega^n,1} = \emptyset$,

iii) $\forall_{n\geq 0} \quad X \subset \bigcup_{\omega^n\in\{0,1\}^n} \operatorname{Int} B_{\omega^n}$,

iv) $\forall_{\omega^n\in\Sigma} \quad B_{\omega^n} \cap X \neq \emptyset$,

v) $\lim_{n\to\infty} \sup_{\omega^n\in\{0,1\}^n} \operatorname{diam} B_{\omega^n} = 0$.

We can easily do this construction for some special Cantor sets (like the usual middle one third Cantor set on a line, contained in the plane). As any two embeddings of Cantor sets in the plane are equivalent with respect to plane homeomorphisms (see [M], chap. 13, theorem 6, p. 93), we get the construction for X. As the distance from X to the boundary of any B_{ω^n} and the distance between boundaries of B_{ω^n} and $B_{\omega^n i}$ is non-zero, we may freely assume that the boundary of all balls B_{ω^n} is C^1.

Let $B^{(n)} = \bigcup_{\omega^n\in\{0,1\}^n} B_{\omega^n}$. By i), they form a decreasing sequence, by iii), iv) and v), we have $X = \bigcap_n B^{(n)}$.

In the course of the proof, we will construct inductively another family of topological closed balls (even euclidean balls, but it is not really necessary) C_{ω^n} and define the contractive map F in such a way that $F((B_{\omega^n})^c) = (C_{\omega^n})^c$. The sets C_{ω^n} will satisfy the properties i), ii), iv) and v) above.

We define the sets $C^{(n)}$ similarly to $B^{(n)}$: it is a decreasing family as well but its limit is only some Cantor subset of X.

Let us note that the construction of the family B_{ω^n} is precisely the point where the proof fails in dimension higher than two. In fact for the Cantor sets in $\mathbb{R}^3$ for which such a construction is still possible, the rest of the proof follows with minor modifications. These Cantor sets can be easily seen as images of a Cantor set on the line $\mathbb{R}^1 \subset \mathbb{R}^3$ under homeomorphisms of $\mathbb{R}^3$.

2.3.2 A geometric lemma

We will use the following lemma:

Lemma 2.6. *Let A, A_0, A_1 be closed topological balls with C^1 boundary, such that A_0 and A_1 are disjoint and contained in* $\operatorname{Int}(A)$. *Let r be a positive real number. Let z, z_0, z_1 be three points such that $z_0, z_1 \in U = B(z, r)$.*

Then, there exists a constant $k(A, A_0, A_1, U, z_0, z_1)$ with the following property: for any Lipschitz homeomorphism $g : \partial A \to \partial B(z, r)$ with Lipschitz constant L and for all sufficiently small r_0, r_1 one can find a map $h : A \setminus \mathrm{Int}(A_0 \cup A_1) \to B(z, r) \setminus \mathrm{Int}(B(z_0, r_0) \cup B(z_1, r_1))$ such that:

a) $h_{|\partial A} = g$,

b) h is a homeomorphism, $h(\partial A_i) = \partial B(z_i, r_i)$,

c) h is Lipschitz and its Lipschitz constant is not greater than

$$L' = L \cdot k(A, A_0, A_1, U, z_0, z_1),$$

d) the Lipschitz constant of $h_{|\partial A_i}$ is not greater than

$$L'' = L \cdot r_i \cdot k(A, A_0, A_1, U, z_0, z_1).$$

Note that the constant $k(A, A_0, A_1, U, z_0, z_1)$ will not change if we rescale the triple $\{U, z_0, z_1\}$ by a similitude or exchange z_0 with z_1.

Proof. As all the possible triples (A, A_0, A_1) are bi-Lipschitz equivalent, it is enough to prove the lemma for $A = B(z, r), A_0 = B(z_0, \rho), A_1 = B(z_1, \rho)$ where ρ is the greatest such number that $B(z_0, 4\rho)$ and $B(z_1, 4\rho)$ are disjoint and contained in $B(z, r)$. Similarly, we may assume that $r = 1, z = (0, 0)$ (the rescaling changes L, L' and L'' by the same multiplicative constant) and that g is orientation preserving, ie.

$$g(\cos\phi, \sin\phi) = (\cos g_0(\phi), \sin g_0(\phi))$$

for some orientation-preserving homeomorphism g_0 of S^1. Note that now $L \geq 1$. We assume $r_0, r_1 < \rho$.

We define h as follows:

- on the annulus $1 \geq |x| \geq 1 - r$ by the formula

  $$h(a\cos\phi, a\sin\phi) =$$
  $$\left(a\cos\left(\frac{1-a}{r}\phi + \frac{a+r-1}{r}g_0(\phi)\right), a\sin\left(\frac{1-a}{r}\phi + \frac{a+r-1}{r}g_0(\phi)\right)\right).$$

- on the set $B((0, 0), 1 - r) \setminus (B(z_0, 2r) \cup B(z_1, 2r))$ as the identity,

- on $B(z_i, 2r)$ by the formula

  $$h(z_i + a \cdot (\cos\phi, \sin\phi)) = z_i + \left(\frac{2r - r_i}{r}a + 2r_i - 2r\right) \cdot (\cos\phi, \sin\phi).$$

This map satisfies a) and b), its Lipschitz constant is not greater than L in the first part, 1 in the second one and 2 in the third one (this gives c)) and its Lipschitz constant on ∂A_i is not greater than r_i/r (we get d)). □

2.3.3 End of the constructions

We are now prepared to construct inductively the sets C_{ω^n} and the contractive homeomorphism F on each set $\mathrm{Int}(C^{(n)}) \setminus \mathrm{Int}(C^{(n+1)})$: we consider the Cantor set X, two points x and y belonging to X and the covering $\{B_{\omega^n}\}$ constructed in the claim above. Without weakening the assumptions, we may choose zeros and ones in our symbolic notation in such a way that $\{x\} = \bigcap B_{0^n}$.

We begin with the definitions of $C_\emptyset = C^{(0)}$ and F on $\mathbb{R}^2 \setminus \mathrm{Int}(C_\emptyset)$. As the Cantor set is perfect, y is not an isolated point, hence we have a sequence (y_ℓ) in X converging to y. For any ℓ, the triple $\{B(y, 2|y_\ell - y|), y, y_\ell\}$ is geometrically identical up to a similitude. We take any Lipschitz homeomorphism f_0 from $B^c_\emptyset$ onto $B^c(y, 2|y_0 - y|)$. We can construct a family of Lipschitz homeomorphisms f_ℓ from $B^c_\emptyset$ onto $B^c(y, 2|y_\ell - y|)$ for all ℓ by $f_\ell(x) = S_\ell \circ f_0(x)$, where S_ℓ is the similitude moving $\{B(y, 2|y_0 - y|), y, y_0\}$ onto $\{B(y, 2|y_\ell - y|), y, y_\ell\}$. The Lipschitz constants of those homeomorphisms are decreasing to zero together with $|y_\ell - y|$, hence for some ℓ they will be smaller than $1/k(B_\emptyset, B_0, B_1, B(y, 2|y_\ell - y|), y, y_\ell) = 1/k(B_\emptyset, B_0, B_1, B(y, 2|y_0 - y|), y, y_0) = c < 1$. Let us denote $C_\emptyset = B(y, 2|y_\ell - y|)$ and define $F = f_\ell$ on $\mathbb{R}^2 \setminus \mathrm{Int}(C_\emptyset)$.

Now we explain how to choose C_0 and C_1 and extend F on $\mathrm{Int}(C_\emptyset) \setminus \mathrm{Int}(C_0 \cup C_1)$. At this step, F is a contraction from the complement of $\mathrm{Int}(B_\emptyset)$ onto the complement of $\mathrm{Int}(C_\emptyset)$ and (by Lemma 2.6) it can be extended to an homeomorphism from the complement of $\mathrm{Int}(B_0 \cup B_1)$ onto the complement of $\mathrm{Int}(B(y, r_0) \cup B(y_\ell, r_1))$ for some small r_0 and r_1. By Lemma 2.6 c) and by our choice of the Lipschitz constant of F in restriction to the boundary $\partial B_\emptyset$, the map F is a contraction.

Moreover by Lemma 2.6 d), for sufficiently small r_0 and r_1 the Lipschitz constant on ∂B_0 and ∂B_1 is arbitrarily small. We may thus choose a point from X very close to y and repeat the procedure, obtaining a contraction on $\mathrm{Int}(B_0) \setminus \mathrm{Int}(B_{00} \cup B_{01})$. Similarly choosing a point sufficiently close to y_n lets us extend our function on $\mathrm{Int}(B_1) \setminus \mathrm{Int}(B_{10} \cup B_{11})$ and so on.

In this inductive procedure we will get a contracting map from the complement of X onto the complement of some Cantor set Y on the plane (the closure of the set of all the points we have chosen on all the stages of the construction). As the points chosen were always belonging to X, the set Y is included in X. We then extend the function F on X by continuity and we are done.

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